Automatica 84 (2017) 159–165

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Wardrop equilibrium on time-varying graphs[☆]

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ARTICLE INFO

Article history:
Received 7 August 2014
Received in revised form 4 April 2017
Accepted 18 July 2017
Available online 17 August 2017

Keywords:
Wardrop equilibrium
Load balancing
Game theory
Graph theory
Switched systems

ABSTRACT

Several problems in transportation and communication networks lead to the notion of Wardrop equilibrium. There exists a large number of algorithms to find Wardrop equilibria, both centralized and distributed. This paper presents a distributed control algorithm, which converges to a Wardrop equilibrium, derived from the algorithm presented in Fischer and Vöcking (2009). The innovation lies in the fact that convergence results are obtained considering that communications occur over time-varying communication graphs, with mild assumptions on the graph connectivity in time (*uniform connectivity*).

1. Introduction

The notion of Wardrop equilibrium, originally introduced for network games when modeling transportation networks with congestion (Wardrop, 1952), can be informally described as "*the journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route*" (Correa & Stier-Moses, 2011).

We refer to a general model of a network of providers working in parallel; each provider is assumed to own a single resource (e.g., processing power, capacity) and there is an infinite stream of infinitely-many arriving users characterized by a demand of resources, expressed in jobs per unit of time. We consider that communications among the providers occur over a time-varying communication network and propose a distributed algorithm, which converges to the Wardrop equilibrium by exploiting information exchange among neighboring providers. The convergence proof relies on recently developed graph-theory concepts, and demonstrates that the algorithm reaches a Wardrop equilibrium under mild hypotheses on the graph connectivity in time.

1.1. State of the art and paper contribution

Wardrop equilibria have been adopted to cope with different types of congested environments, e.g.: routing in road traffic networks (Chiou, 2010; Como, Savla, Acemoglu, Dahleh, & Frazzoli, 2013), traffic engineering in communication networks (Fischer, Kammenhuber, & Feldmann, 2006), load balancing in distributed computational grids (Grosu & Chronopoulos, 2004) and in wired and wireless networks (Anselmi, Ayesta, & Wierman, 2011; Oddi & Pietrabissa, 2013; Oddi, Pietrabissa, Delli Priscoli, & Suraci, 2014).

The classical problem formulation (Barth, Bournez, Boussaton, & Cohen, 2008) considers a congested network represented as a graph with nodes and edges. Each edge is associated to a non-decreasing latency function of the traffic sent over it. The network serves several commodities, characterized by a given amount of traffic to be routed from a source to a destination. The jobs vector is the amount of traffic that is allocated for each feasible path, for each commodity. A jobs vector in which, for all commodities, the latencies of all used paths are equal is called Wardrop equilibrium, which can be computed by centralized algorithms in polynomial time (Beckmann, McGuire, & Winsten, 1959). This paper focuses on dynamic and distributed algorithms able to compute the Wardrop equilibrium.

In Borkar and Kumar (2003), a distributed and asynchronous routing algorithm is proposed, which relies on an estimation of the latencies of all the paths and on a reinforcement learning algorithm to update the transmission probabilities towards the different paths. In Fischer, Olbrich, and Vöcking (2008), the authors develop a round-based distributed algorithm with a finite number of agents. Each agent is responsible for one commodity and has a set of admissible paths among which it may distribute its traffic. A bulletin board is assumed to be available, on which the traffic assignments are updated at every round. With a similar

[☆] The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Claudio De Persis under the direction of Editor Ian R. Petersen.

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http://dx.doi.org/10.1016/j.automatica.2017.07.021
0005-1098/

bulletin board scenario, in Fischer and Vöcking (2009) a routing algorithm is presented, based on rules for each agent to 'sample' a different path and to 'migrate' to a different path: if the latency of the sampled path is smaller than the latency of the current one, the agent migrates to the new path with a given probability. Convergence results are given both when the agents base their decisions on up-to-date information, and when the information are "stale" (i.e., considering delays in the bulletin board update). In Fischer, Räcke, and Vöcking (2010), a round-based version of the algorithm in Fischer and Vöcking (2009) is proposed.

In Anselmi et al. (2011) the authors consider the application of the Wardrop equilibrium to the *load balancing game* with complete communications among providers.

Up to the authors' knowledge, no works in the literature consider communication constraints posed by the network topology. The objective of this paper is to develop an algorithm which converges to a Wardrop equilibrium in a time-varying communication-constrained environment. This paper adopts the problem formulation of Anselmi et al. (2011) and extends the algorithm proposed in Fischer and Vöcking (2009). Note that, in the proposed model, there is no need to consider the case of stale information, since only information exchange between neighbors is assumed: no multi-hop communication is necessary and there is no centralized bulletin board to be kept updated.

The paper is organized as follows: Section 2 describes the problem model; Section 3 presents the convergence results; Section 4 draws the conclusions.

2. Preliminaries and problem model

Standard notation is used through the paper, with bold letters to denote vectors. The operator $|\cdot|$ denotes the cardinality of a set. The system is considered for time $t \geq 0$.

2.1. Graph theory and time-varying connectivity

Let $G = (V, E, w)$ be a weighted graph, where V is the finite set of vertices or nodes, $E \subseteq V \times V$ is the set of edges or links and $w : E \to (0, M]$, $0 < M < +\infty$, is a weights map associating each link $(p, q) \in E$ to a positive real value, denoted with w_{pq}. The adjacency matrix is defined as $A = \left\{a_{pq}\right\}_{p,q\in V} \in \{0, 1\}^{|V|\times|V|}$, with $a_{pq} = 1$ if $(p, q) \in E$, $a_{pq} = 0$ otherwise. The weighted adjacency matrix is defined as $A^w = \left\{a^w_{pq}\right\}_{p,q\in V} \in [0, M]^{|V|\times|V|}$, with $a^w_{pq} = a_{pq} w_{pq}$.

A graph G is *undirected* if A is symmetric and *directed* otherwise. This paper considers undirected graphs.

A node $q \in V$ is an adjacent node, or *neighbor*, of node $p \in V$ if $a_{pq} = 1$. Let N_p be the set of neighbors of p: $N_p := \left\{q \in V | a_{pq} = 1\right\}$, $p \in V$. The *degree* of node p is the number $\left|N_p\right|$ of its neighbors. The degree of the graph nodes roughly indicates "how much" the graph is connected.

A *path* is a sequence of distinct edges which connect a sequence of adjacent nodes. Let $\mathcal{P}$ be the set of the paths in G. Two nodes $p, q \in V$ are *connected* if there is a path $\pi \in \mathcal{P}$ from p to q, they are disconnected otherwise.

An undirected graph G is *connected* if every pair of nodes is connected. If the graph is not connected, m connected sub-graphs, referred to as *connected components*, can be identified, with $1 < m \leq |V|$ (if $m = 1$ the graph is connected, if $m = |V|$ the graph is completely disconnected with $E = \varnothing$). A graph is *complete* if $a_{pq} = 1, \forall p, q$.

Let $\mathcal{G}(V) := \{G_s = (V, E_s, w_s) \,|E_s \subseteq V \times V, w_s : E_s \to (0, M_s],$ with $0 < M_s < +\infty\}$ be the set of all the possible graphs over the set of nodes V. *Time-varying graphs* can be defined as follows (Nicosia, Tang, Musolesi, Russo, Mascolo, & Latora, 2012):

Definition 1. A time-varying graph $G(t)$ is characterized by edges that appear and disappear over time, and can be described as an ordered sequence of graphs $G_s \in \mathcal{G}(V)$, $s = 0, 1, \ldots$, characterized by a set of edges E_s defined over a non-null time interval $T_s = [t_s, t_{s+1})$, with $t_s < t_{s+1}$ and $t_0 = 0$, and by a weights map w_s. The instants t_s are such that a finite number of instants occur in a limited time interval.

The adjacency matrix, the weighted adjacency matrix, the weights and the neighbor sets are time-varying and are denoted with $A(t) = A_s = \left\{a_{pq,s}\right\}_{p,q\in V}$, $A^w(t) = A^w_s = \left\{a^w_{pq,s}\right\}_{p,q\in V}$, $w_{pq,s}$, $p, q \in V$ and$N_p(t) = N_{p,s}$, $\forall p \in V$, respectively, $\forall t \in T_s$, $s = 0, 1, \ldots$. ■

The notion of connectivity can be extended to time-varying graphs. Preliminarily, the φ-graph definition is needed:

Definition 2 (*Moreau, 2004*)**.** Let A^w be a weighted adjacency matrix with weights map $w : E \to (0, M]$, $0 < M < \infty$. Given a threshold $\varphi \in [0, M)$, the φ-graph associated to A^w is a graph where an edge from a given node p to a given node q exists if and only if $a^w_{pq} > \varphi$. ■

This paper focuses on the following connectivity definition (Moreau, 2004, 2005), often denoted in the literature as *uniform connectivity* (Delli Priscoli, Isidori, Marconi, & Pietrabissa, 2015; Wieland, Sepulchre, & Allgöwer, 2011):

Definition 3. A time-varying graph $G(t)$, with weighted adjacency matrix $A^w(t)$, is uniformly connected if there exist a threshold $\varphi \geq 0$ and a constant $\mathcal{T} > 0$ such that the φ-graph associated to the weighted adjacency matrix

$$\mathcal{A}^w(t) = \frac{1}{\mathcal{T}} \int_t^{t+\mathcal{T}} A^w(z)\, dz \tag{1}$$

is connected for all $t \geq 0$. ■

2.2. Problem model and Wardrop equilibrium

In this section, a general model is described, usually applied for load balancing problems (Anselmi et al., 2011). The set V of nodes of the graph G is regarded as a network of providers working in parallel. Each provider is assumed to own a single separate resource and there is an infinite stream of infinitely-many arriving users, or *agents*, characterized by a *demand* of $\lambda > 0$, expressed in jobs per unit of time. In the following, we refer to the terms *provider* and *node* as synonyms.

According to the Wardrop model, each agent carries an infinitesimally small amount of the demand. The *jobs rate* or *population share* x_p is the amount of jobs per unit of time assigned to provider $p \in V$. The *jobs vector* $\boldsymbol{x} = \left(x_p\right)_{p\in V}$ is *feasible* if $x_p \geq 0, \forall p \in V$ and $\sum_{p\in V} x_p = \lambda$. Let $\mathcal{X}$ be the bounded set of the feasible jobs vectors:

$$\mathcal{X} := \left\{ \boldsymbol{x} = \left(x_p\right)_{p\in V} \Big| \sum_{p\in V} x_p = \lambda, x_p \geq 0, \forall p \in V \right\}. \tag{2}$$

A metric of interest is the mean time it takes for a job to be served by a provider p. This quantity is the *latency* of provider p, which is a non-negative function $l_p\left(x_p\right)$ of the amount of jobs per unit of time to be handled by provider p. The shape of the latency functions depends on the considered application. One strength of the proposed approach is that there is no need of explicitly modeling the latency functions, which can also be different for each provider $p \in V$ (*heterogeneous providers*). The latency functions are only assumed to have the following properties: