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Consensus dynamics with arbitrary sign-preserving nonlinearities☆

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ABSTRACT

This paper studies consensus problems for multi-agent systems defined on directed graphs where the consensus dynamics involves general nonlinear and discontinuous functions. Sufficient conditions, only involving basic properties of the nonlinear functions and the topology of the underlying graph, are derived for the agents to converge to consensus.

1. Introduction

Nonlinear agreement protocols have recently attracted the attention of many researchers. They may arise due to the nature of the controller, see e.g. Jafarian and De Persis (2015) and Saber and Murray (2003), or may describe the physical coupling existing in the network, see e.g. Bürger, Zelazo, and Allgöwer (2014) and Monshizadeh and De Persis (2015). In this paper, we consider a general nonlinear consensus protocol. The topology among the agents is assumed to be a directed graph containing a directed spanning tree, which for the linear consensus protocol is known to be a sufficient and necessary condition for reaching state consensus.

Previous work related to this paper can be divided into two categories, depending on whether the dynamical systems are continuous or not. For the case of continuous dynamical systems, closely related to this paper are Lin, Francis, and Maggiore (2007) and Papachristodoulou, Jadbabaie, and Münz (2010). In Papachristodoulou et al. (2010), a general first-order consensus protocol with a continuous nonlinear function is considered for the case that there is delay in the communication. In Lin et al. (2007), the authors considered a nonlinear consensus protocol with Lipschitz continuous functions, under a switching topology. For the case of discontinuous dynamical system, Cortés (2006) is one of the major motivations of this paper. Nonlinearities of the form of sign functions were considered in Cortés (2006), where the notion of Filippov solutions is employed. However, in order to guarantee asymptotic consensus of the second network protocol in Section 4 of Cortés (2006), further conditions turn out to be necessary. This is formulated as the main result in Section 3.2 of this paper. In De Persis and Frasca (2013), the authors considered a similar control protocol as in Cortés (2006) in a hybrid dynamical systems framework with a self-triggered communication policy, which avoids the notion of Filippov solutions. In addition, in De Persis and Frasca (2013) practical consensus is considered, that is, consensus within a predefined margin. The results presented in Cortés (2006) and De Persis and Frasca (2013) are restricted to undirected graphs. In Dimarogonas and Johansson (2010), the authors considered quantized communication protocols within the framework of hybrid dynamical systems, without using the notion of Filippov solutions.

The contribution of this paper is to provide a uniform framework to analyze the convergence towards consensus of a first-order consensus protocol for a very general class of discontinuous nonlinear functions, under the weakest fixed topology assumption, i.e., a directed graph containing a directed spanning tree. The analysis is conducted with the notion of Filippov solutions, and generalizes and corrects the second network consensus protocol in Cortés (2006).

The structure of the paper is as follows. In Section 2, we introduce some terminology and notation in the context of graph theory and stability analysis of discontinuous dynamical systems. The main results are presented in Theorems 7 and 19 in Section 3. The general problem is introduced in Section 3.1, whereafter in Sections 3.2 and 3.3 two important subcases are considered, which are then combined in Section 3.4. In Section 4 we study error dynamics corresponding to the systems considered in Section 3.2,

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and provide sufficient conditions for the equivalence between the convergence of the error to zero and the convergence of the original states to consensus. Conclusions follow in Section 5.

2. Preliminaries and notations

In this section we briefly review some notions from graph theory, and give some definitions and notation regarding Filippov solutions.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a weighted digraph with node set $\mathcal{V} = \{v_1, \ldots, v_n\}$, edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and weighted adjacency matrix $A = [a_{ij}]$ with nonnegative adjacency elements a_{ij}. An edge of $\mathcal{G}$ is denoted by $e_{ij} := (v_i, v_j)$ and we write $\mathcal{I} = \{1, 2, \ldots, n\}$. The adjacency elements a_{ij} are associated with the edges of the graph such that $a_{ij} > 0$ if and only if $e_{ji} \in \mathcal{E}$, while $a_{ii} = 0$ for all $i \in \mathcal{I}$. For undirected graphs $A = A^\top$.

The set of neighbors of node v_i is denoted by $N_i := \{v_j \in \mathcal{V} : e_{ji} \in \mathcal{E}\}$. For each node v_i, its in-degree and out-degree are defined as

$$\deg_{\text{in}}(v_i) = \sum_{j=1}^{n} a_{ij}, \qquad \deg_{\text{out}}(v_i) = \sum_{j=1}^{n} a_{ji}.$$

The degree matrix of the digraph $\mathcal{G}$ is a diagonal matrix Δ where $\Delta_{ii} = \deg_{\text{in}}(v_i)$. The *graph Laplacian* is defined as $L = \Delta - A$ and satisfies $L\mathbb{1} = 0$, where $\mathbb{1}$ is the n-vector containing only ones. We say that a node v_i is *balanced* if its in-degree and out-degree are equal. The graph $\mathcal{G}$ is called balanced if all of its nodes are balanced or, equivalently, if $\mathbb{1}^\top L = 0$.

A directed path from node v_i to node v_j is a chain of edges from $\mathcal{E}$ such that the first edge starts from v_i, the last edge ends at v_j and every edge in between starts where the previous edge ends. If for every two nodes v_i and v_j there is a directed path from v_i to v_j, then the graph $\mathcal{G}$ is called *strongly connected*. A subgraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', A')$ of $\mathcal{G}$ is called a *directed spanning tree* for $\mathcal{G}$ if $\mathcal{V}' = \mathcal{V}$, $\mathcal{E}' \subseteq \mathcal{E}$, and for every node $v_i \in \mathcal{V}'$ there is exactly one node v_j such that $e_{ji} \in \mathcal{E}'$, except for one node, which is called the root of the spanning tree. Furthermore, we call a node $v \in \mathcal{V}$ a *root* of $\mathcal{G}$ if there is a directed spanning tree for $\mathcal{G}$ with v as a root. In other words, if v is a root of $\mathcal{G}$, then there is a directed path from v to every other node in the graph. A digraph $\mathcal{G}$ is called *weakly connected* if $\mathcal{G}^o$ is connected, where $\mathcal{G}^o$ is the undirected graph obtained from $\mathcal{G}$ by ignoring the orientation of the edges.

The multi-dimensional saturation function *sat* and sign function *sign* are defined as follows. For any $x \in \mathbb{R}^n$,

$$sat(x; u^-, u^+)_i = \begin{cases} u_i^- & \text{if } x_i < u_i^-, \\ x_i & \text{if } x_i \in [u_i^-, u_i^+], \\ u_i^+ & \text{if } x_i > u_i^+ \end{cases} \quad i \in \mathcal{I}, \tag{1}$$

$$sign(x)_i = \begin{cases} -1 & \text{if } x_i < 0, \\ 0 & \text{if } x_i = 0, \\ 1 & \text{if } x_i > 0, \end{cases} \quad i \in \mathcal{I}, \tag{2}$$

where u^- and u^+ are n-vectors containing the lower and upper bounds respectively.

With $\mathbb{R}_-$, $\mathbb{R}_+$ and $\mathbb{R}_{\geqslant 0}$ we denote the sets of negative, positive and nonnegative real numbers respectively. The vectors $e_1, e_2, \ldots, e_n$ denote the canonical basis of $\mathbb{R}^n$. The ith row and jth column of a matrix M are denoted by $M_{i\cdot}$ and $M_{\cdot j}$ respectively. For the empty set, we adopt the convention that $\max \emptyset = -\infty$.

In the rest of this section we give some definitions and notations regarding Filippov solutions (see, e.g., Cortés (2008)). Let F be a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, and let $2^{\mathbb{R}^n}$ denote the collection of all subsets of $\mathbb{R}^n$. The map F is *essentially bounded* if there is a bound B such that $\|F(x)\|_2 < B$ for almost every $x \in \mathbb{R}^n$. The map F is *locally essentially bounded* if the restriction of F to every compact subset of $\mathbb{R}^n$ is essentially bounded. The *Filippov set-valued map* of F, denoted $\mathcal{F}[F] : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, is given as

$$\mathcal{F}[F](x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \overline{\text{co}}\{F(B(x,\delta) \setminus S)\}, \tag{3}$$

where $B(x, \delta)$ is the open ball centered at x with radius $\delta > 0$, $S \subset \mathbb{R}^n$, μ denotes the Lebesgue measure and $\overline{\text{co}}$ denotes the convex closure. The zero measure set S is arbitrarily chosen. Hence, the set $\mathcal{F}[F](x)$ is independent of the value of $F(x)$. If F is continuous at x, then $\mathcal{F}[F](x) = \{F(x)\}$. A *Filippov solution* of the differential equation $\dot{x}(t) = F(x(t))$ on $[0, t_1] \subset \mathbb{R}$ is an absolutely continuous function $x : [0, t_1] \to \mathbb{R}^n$ that satisfies the differential inclusion

$$\dot{x}(t) \in \mathcal{F}[F](x(t)) \tag{4}$$

for almost all $t \in [0, t_1]$. Let f be a map from $\mathbb{R}^n$ to $\mathbb{R}$. We use the same definition of regular function as in Clarke (1990) and recall that convex functions are regular. If $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, then its *generalized gradient* $\partial f : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is defined by

$$\partial f(x) := \text{co}\{\lim_{i\to\infty} \nabla f(x_i) \mid x_i \to x, x_i \notin S \cup \Omega_f\}, \tag{5}$$

where ∇ denotes the gradient operator, $\Omega_f \subset \mathbb{R}^n$ denotes the set of points where f is not differentiable, and $S \subset \mathbb{R}^n$ is an arbitrary set of measure zero. ($\partial f(x)$ is independent of the choice of S Clarke (1990).) Given a set-valued map $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, the *set-valued Lie derivative* $\tilde{\mathcal{L}}_{\mathcal{F}} f : \mathbb{R}^n \to 2^{\mathbb{R}}$ of a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ with respect to $\mathcal{F}$ at x is defined as

$$\tilde{\mathcal{L}}_{\mathcal{F}} f(x) := \{a \mid \exists v \in \mathcal{F}(x) \text{ s.t. } \zeta^\top v = a \; \forall \zeta \in \partial f(x)\}.$$

A Filippov solution $t \mapsto x(t)$ is *maximal* if it cannot be extended forward in time. Since the Filippov solutions of a discontinuous system (4) are not necessarily unique, we need to specify two types of invariant sets. A set $\mathcal{R} \subset \mathbb{R}^n$ is called *weakly invariant* for (4) if, for each $x_0 \in \mathcal{R}$, at least one maximal solution of (4) with initial condition x_0 is contained in $\mathcal{R}$. Similarly, $\mathcal{R} \subset \mathbb{R}^n$ is called *strongly invariant* for (4) if, for each $x_0 \in \mathcal{R}$, every maximal solution of (4) with initial condition x_0 is contained in $\mathcal{R}$. For more details, see Cortés (2008).

3. Main results

3.1. Problem formulation

In this work we consider a network of n agents, who communicate according to a communication topology given by a weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. In this network, agent i receives information from agent j if and only if there is an edge from node v_j to node v_i in the graph $\mathcal{G}$. We denote the state of agent i at time t as $x_i(t) \in \mathbb{R}$, and consider the following dynamics for agent i

$$\dot{x}_i = f_i\left(\sum_{j=1}^{n} a_{ij} g_{ij}(x_j - x_i)\right) =: h_i(x), \tag{6}$$

where f_i and g_{ij} are functions, a_{ij} are the elements of the adjacency matrix A.

Each function f_i describes how agent i handles incoming information, while g_{ij} are concerned with the flow of information along the edges of the graph $\mathcal{G}$. All these functions are nonlinear and may have discontinuities, but we will use the concept of sign-preserving functions.

Definition 1. We say that a function $\varphi : \mathbb{R} \to \mathbb{R}$ is *sign preserving* if $\varphi(0) = 0$ and for each $y \in \mathbb{R} \setminus \{0\}$ we have both $y\varphi(y) > 0$ and $\min y\mathcal{F}[\varphi](y) > 0$.