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Zeros of nonlinear systems with input invariances[☆]

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ABSTRACT

A nonlinear system possesses an invariance with respect to a set of transformations if its output dynamics remain invariant when transforming the input, and adjusting the initial condition accordingly. Most research has focused on invariances with respect to time-independent pointwise transformations like translational-invariance ($u(t) \mapsto u(t) + p$, $p \in \mathbb{R}$) or scale-invariance ($u(t) \mapsto pu(t)$, $p \in \mathbb{R}_{>0}$). In this article, we introduce the concept of s_0-invariances with respect to continuous input transformations exponentially growing/decaying over time. We show that s_0-invariant systems not only encompass linear time-invariant (LTI) systems with transfer functions having an irreducible zero at $s_0 \in \mathbb{R}$, but also that the input/output relationship of nonlinear s_0-invariant systems possesses properties well known from their linear counterparts. Furthermore, we extend the concept of s_0-invariances to second- and higher-order s_0-invariances, corresponding to invariances with respect to transformations of the time-derivatives of the input, and encompassing LTI systems with zeros of multiplicity two or higher. Finally, we show that nth-order 0-invariant systems realize – under mild conditions – nth-order nonlinear differential operators: when excited by an input of a characteristic functional form, the system's output converges to a constant value only depending on the nth (nonlinear) derivative of the input.

1. Introduction

Systems invariant with respect to a set of pointwise input transformations (see e.g. Adler, Mayo, & Alon, 2014; Goentoro, Shoval, Kirschner, & Alon, 2009; Hironaka & Morishita, 2014; Shoval, Alon, & Sontag, 2011; Shoval et al., 2010) show the same output dynamics when applying a transformation to the systems' input, and adjusting the initial conditions appropriately (see Section 3 for precise definitions). For example, linear time-invariant (LTI) systems with a zero at the origin are *translational invariant*, that is, invariant with respect to translations $u(t) \mapsto u(t) + p$ of the input, with $p \in \mathbb{R}$. Similarly, *scale-invariance* (also referred to as fold-change detection Shoval et al., 2010) is defined with respect to geometric scaling $u(t) \mapsto pu(t)$ of the input, with $p \in \mathbb{R}_{>0}$. In the context of invariant systems, the major result in Shoval et al. (2011) is of specific importance, showing that nonlinear systems are invariant with respect to a certain set of input transformations if and only if they are *equivariant* with respect to the same transformations (see Section 3 for details). Different to invariance, equivariance is a "memoryless" structural property only depending on the current state and input of the system; that a given system is equivariant is, thus, typically easier to prove.

Recently, we have shown that – under mild conditions – invariant systems realize first-order nonlinear differential operators (Lang & Sontag, 2016). That is, there exists a set of characteristic inputs for which the output of an equivariant system remains constant (in general nonzero) when initialized appropriately. Importantly, the constant value of the output only depends on the (nonlinear) derivative of the characteristic input, with the functional form of the derivative defined by the invariance itself. For example, translational invariant systems can realize the differential operator $\frac{d}{dt}$ (i.e., $\bar{y}^* = \alpha(\frac{d}{dt}u(t))$, with u a characteristic input, $\bar{y}^*$ the constant output, and α some nonlinear map) in agreement with the known property that the output of Hurwitz LTI systems with a zero at the origin excited by ramps converge to constant values proportional to the slope of the ramp. Similarly, scale-invariant systems can realize the nonlinear differential

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operator $\frac{d}{dt}\log$ (i.e., $\bar{y}^* = \alpha\left(\frac{d}{dt}\log(u(t))\right)$), with the characteristic inputs given by exponential functions (Lang & Sontag, 2016).

In this article, we introduce two mutually compatible generalizations of invariance: (i) for any given $s_0 \in \mathbb{R}$, s_0-invariant systems are invariant with respect to continuous input transformations exponentially growing/decaying over time, and comprise LTI systems with a zero at s_0; and (ii) second-order s_0-invariant systems are invariant with respect to transformations of the time-derivative of the input, and comprise LTI systems possessing zeros with multiplicity two. Additionally, we show how the latter can be generalized to arbitrary-order invariances. For each of the two generalizations of invariance, we derive the corresponding generalized equivariances, that is, provide a possibility to structurally test if a given system possesses a generalized invariance without having to consider state trajectories or past inputs. Finally, based on the definition of a characteristic model of an s_0-invariant system – a concept related to pole-zero cancellation of LTI systems – we extend our previous results on systems realizing differential operators (Lang & Sontag, 2016) by showing that higher-order 0-invariant systems can realize higher-order differential operators.

The rest of this article is organized as follows. In Section 2, we briefly recapitulate dynamic properties of LTI systems with zeros of arbitrary multiplicity. In Section 3, we provide our definitions of first-order s_0-invariance and s_0-equivariance, and prove their equivalence under mild assumptions. In Section 6, we introduce the notion of second-order s_0-equivariance and invariance. Finally, based on the characteristic model of an s_0-invariant system (Section 4), we define in Sections 5 and 7 systems realizing first- and higher-order differential operators, and establish their close relationship to 0-invariant systems.

2. Zeros of linear time-invariant systems

Consider a single-input, single-output LTI system given by the ordinary differential equations (ODEs)

$$\frac{d}{dt}z(t) = Az(t) + bu(t), \qquad z(0) = \bar{z} \tag{1a}$$

$$y(t) = c^T z(t). \tag{1b}$$

with state $z(t) \in \mathbb{R}^n$, piecewise-continuous input $u(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$, system matrix $A \in \mathbb{R}^{n\times n}$, and input and output vectors $b, c \in \mathbb{R}^{n\times 1}$.

An LTI system (1) has a zero at $s_0 \in \mathbb{C}$ if $\det(M(s_0)) = 0$, with $M(s) = \begin{bmatrix} A - sI & b \\ -c^T & 0 \end{bmatrix}$ (see e.g. Brockett, 1965). In the following, we assume that the system (1) is controllable and observable. Since, in this article, we are only interested in real zeros, we furthermore assume $s_0 \in \mathbb{R}$. Then, a necessary and sufficient condition for the system to have a zero at s_0 is that the transfer function $G(s) = -c^T(A - sI)^{-1}b$ evaluates to zero at $s = s_0$. The nullspace of $M(s_0)$ is spanned by the vector $(\bar{z}^1_{s_0}, 1)^T$, with $\bar{z}^1_{s_0} = -(A - s_0 I)^{-1}b$. This implies that $\left(pu^1_{s_0}, p\bar{z}^1_{s_0}\right)$, with $u^1_{s_0}(t) = e^{s_0 t}$ and $p \in \mathbb{R}$, zeros the output (see e.g. Isidori, 1995, p. 162ff), that is, when (1) is initialized at $p\bar{z}^1_{s_0}$ and excited by $pu^1_{s_0}$, the output remains zero.

More generally, if the system possesses a zero $s_0 \in \mathbb{R}$ with multiplicity $m_{s_0} \geq 1$, i.e. if $c^T(A - sI)^{-l}b = 0$ for all $l = 1, \ldots, m_{s_0}$, the corresponding inputs and initial conditions zeroing the output are given by (multiples of) $u^l_{s_0}(t) = t^{l-1}e^{s_0 t}$ and $\bar{z}^l_{s_0} = -(l - 1)!(A - s_0 I)^{-l}b$. Due to the superposition principle of linear systems, for an LTI system with n_s distinct zeros s^i_0, $i = 1, \ldots, n_s$, with multiplicities $m_{s^i_0}$, the set $S_0 = \{(u^l_{s^i_0}, \bar{z}^l_{s^i_0})\}_{l=1,\ldots,m_{s^i_0}, i=1,\ldots,n_s}$ spans a vector space of inputs and initial conditions zeroing the output.

The superposition principle has another important consequence: Consider an LTI system initialized at any state $\bar{z} \in \mathbb{R}^n$ and excited by any input $u \in \mathcal{U}$, and let $(u_0, \bar{z}_0) \in \operatorname{span}(S_0)$. Then, the output of the system is *invariant* with respect to the mapping $(u, \bar{z}) \mapsto (u+u_0, \bar{z}+\bar{z}_0)$, that is, $c^T\xi(t, \bar{z}, u) = c^T\xi(t, \bar{z}+\bar{z}_0, u+u_0)$, with $\xi(t, \bar{z}, u) = z(t)$ the solution of (1) for the initial condition $\bar{z}$ and the input u. We can reformulate this property as follows: First, let $\pi_p : \mathbb{R} \to \mathbb{R}$, $\pi_p(\bar{u}) = \bar{u} + p$, describe a set of input transformations for $p \in \mathbb{R}$. Assume that the LTI system has a zero at s_0 with multiplicity m_{s_0}. Then, for every $l = 1, \ldots, m_{s^i_0}$, $\bar{z} \in \mathbb{R}$ and $p \in \mathbb{R}$, there exists a $\bar{z}' \in \mathbb{R}$ such that

$$c^T\xi(t, \bar{z}, u) = c^T\xi(t, \bar{z}', t \mapsto \pi_{pt^{l-1}\exp(s_0 t)}(u(t))), \tag{2}$$

for all $u \in U$ and $t \geq 0$. Note, that (2) follows from our previous analysis, with $\bar{z}' = \bar{z} - p(l - 1)!(A - s_0 I)^{-l}b$.

LTI systems with a zero at the origin with multiplicity m_0 possess another interesting property: when excited by $u(t) = \sum_{l=0}^{m_0} k_l t^l$, there exists an initial condition $\bar{z}^*$ such that the output remains constant, i.e. such that $c^T\xi(t, \bar{z}^*, u) = \bar{y}^* = -m_0! c^T A^{-m_0-1} b k_{m_0}$. Notably, $\bar{y}^*$ only depends on the coefficient of the monomial of u with degree m_0. Thus, we can equivalently write $\bar{y}^* = -c^T A^{-m_0-1} b \frac{d^{m_0}}{dt^{m_0}} u(t)$, i.e. the output of the system excited by u and initialized at $\bar{z}^*$ is proportional to the m_0th time derivative of the input. If, additionally, the system is Hurwitz, the output converges to $\bar{y}^*$ for any initial condition.

3. First-order s_0-invariance and s_0-equivariance

Throughout the rest of this article, we consider nonlinear systems given by ODEs of the form

$$\frac{d}{dt}z(t) = f(z(t), u(t)), \qquad z(0) = \bar{z} \tag{3a}$$

$$y(t) = h(z(t)). \tag{3b}$$

The vector $z(t) \in Z \subseteq \mathbb{R}^n$ represents the state of the system at time $t \in \mathbb{R}_{\geq 0}$, and $u \in \mathcal{U} \subseteq \mathcal{PC}(\mathbb{R}_{\geq 0}, U)$ a piecewise-continuous (external) input, with $u : \mathbb{R}_{\geq 0} \to U \subseteq \mathbb{R}$. The dynamics are given by the vector field $f : Z \times U \to \mathbb{R}^n$, the initial conditions by $\bar{z} \in Z$, and the output by $y(t) \in \mathbb{R}$, with $h : Z \to \mathbb{R}$. We assume that f and h are analytic, and that for each initial condition $\bar{z} \in Z$ and each input $u \in \mathcal{U}$ there exists a unique, piecewise differentiable and continuous solution of Eq. (3), which we denote by $\xi : \mathbb{R}_{\geq 0} \times Z \times \mathcal{U} \to Z$, $\xi(t, \bar{z}, u) = z(t)$.

In the previous section, we have shown that the output of an LTI system is invariant with respect to the input transformations $\pi_p : U \to U$, $p \in \mathbb{R}$, corresponding to translations of the input (2). In the following, we define an equivalent property for nonlinear systems – referred to as s_0-*invariance* – with respect to input transformations not necessarily corresponding to translations. Different to previous work (Shoval et al., 2011), we restrict ourselves to input transformations forming a one-parameter Lie group under function composition $\circ$ as defined in Bluman and Kumei (1989). This implies that we can parametrize the input transformations $\mathcal{P} = \{\pi_p : U \to U\}_p$ by a parameter $p \in P \subseteq \mathbb{R}$ such that π_p is differentiable in U and analytic in P (Bluman & Kumei, 1989, p. 34). Furthermore, by the first fundamental theorem of Lie (Bluman & Kumei, 1989, p. 37), π_p can always be parametrized such that $P = \mathbb{R}$, such that the law of composition becomes additive ($\pi_{p_1} \circ \pi_{p_2} = \pi_{p_1+p_2}$), and such that $p = 0$ corresponds to the identity transformation ($\pi_0(\bar{u}) = \bar{u}$ for all $\bar{u} \in U$). In the following, we assume that every Lie group is parametrized as described above.

Definition 1 (*First-order s_0-invariance*)**.** Consider the system (3) and a one-parameter Lie group of input transformations $\mathcal{P} = \{\pi_p : U \to U\}_{p\in\mathbb{R}}$. Then, the system is *first-order s_0-invariant* with