Automatica 81 (2017) 22–29

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Lyapunov stability for piecewise affine systems via cone-copositivity ☆

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ARTICLE INFO

Article history:
Received 24 August 2015
Received in revised form
8 November 2016
Accepted 20 February 2017
Available online 20 April 2017

Keywords:
Piecewise affine systems
Piecewise quadratic Lyapunov functions
Cone-copositivity
Asymptotic stability
Multi-agent system
Opinion dynamics
Saturated systems

ABSTRACT

Cone-copositive piecewise quadratic Lyapunov functions (PWQ-LFs) for the stability analysis of continuous-time piecewise affine (PWA) systems are proposed. The state space is assumed to be partitioned into a finite number of convex, possibly unbounded, polyhedra. Preliminary conditions on PWQ functions for their sign in the polyhedra and continuity over the common boundaries are provided. The sign of each quadratic function is studied by means of cone-constrained matrix inequalities which are translated into linear matrix inequalities (LMIs) via cone-copositivity. The continuity is guaranteed by adding equality constraints over the polyhedra intersections. An asymptotic stability result for PWA systems is then obtained by finding a continuous PWQ-LF through the solution of a set of constrained LMIs. The effectiveness of the proposed approach is shown by analyzing an opinion dynamics model and two saturating control systems.

1. Introduction

Piecewise affine (PWA) systems are characterized by a set of state-dependent switching affine subsystems defined over a state space partitioned into convex polyhedra (Sontag, 1981). There exist numerous important applications which involve PWA systems. At least they can be employed to approximate nonlinear systems and are shown to be equivalent to several classes of hybrid systems (Heemels, De Schutter, & Bemporad, 2001).

The stability analysis of PWA systems is a difficult issue due to their hybrid nature. A classical sufficient condition is the quadratic stability (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994; Eren, Shenb, & Camlibel, 2014) which is however known to be conservative. Different approaches have been investigated in the last years with the aim of obtaining less conservative results. Among others, the multiple Lyapunov function approach, i.e., to combine Lyapunov functions defined over different regions of the state space, has been proposed, see Lin and Antsaklis (2009). In particular, piecewise quadratic Lyapunov functions (PWQ-LFs) obtained by patching together *quadratic forms* (for the regions containing the origin) and *quadratic functions* (for the regions which do not contain the origin), have been widely investigated starting from the seminal work (Johansson, 2003). In this framework the stability conditions are typically formulated in terms of constrained inequalities which can be solved by means of a set of linear matrix inequalities (LMIs) by applying the $\mathcal{S}$-procedure (Lin & Antsaklis, 2009). Unfortunately the $\mathcal{S}$-procedure is lossy in general. Several variants of this technique have been proposed in the more recent literature including sliding modes (Samadi & Rodrigues, 2011), attraction domain estimation (Li & Lin, 2015), and relaxed LMIs for discrete-time PWA systems (Hovd & Olaru, 2013).

In Iervolino and Vasca (2014) a PWQ-LF approach suitable for Lur'e systems with slab partitions is proposed, however the results therein cannot be directly extended to PWA systems with more general polyhedral partitions of the state space. In Iervolino, Vasca, and Iannelli (2015) conewise linear systems were considered, which excluded the presence of bounded polyhedra in the state space partition. In this paper we propose a new PWQ approach for continuous-time PWA systems where the PWQ-LF, differently from the other approaches, is obtained by suitably combining *quadratic functions* for *all* regions of the state space partition. The stability conditions are expressed in terms of cone-constrained inequalities which are translated into LMIs by formulating a cone-copositive problem. The copositive programming for a given matrix analyzed in Bundfuss and Dür (2008) and Sponsel, Bundfuss, and Dür (2012) is here exploited

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Jamal Daafouz under the direction of Editor Richard Middleton.

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http://dx.doi.org/10.1016/j.automatica.2017.03.011

with a more challenging perspective. Indeed our problem consists in finding a set of cone-copositive matrices that define a PWQ-LF and whose entries are degrees of freedom for the stability problem. The approach is shown to be effective for the stability analysis of opinion dynamics (Yang, Dimarogonas, & Hu, 2014) and saturated control systems (Eghbal, Pariz, & Karimpour, 2013; Milani, 2004).

The paper is organized as follows. In Section 2 some preliminary definitions and concepts are recalled. The sign analysis for a PWQ function is considered in Section 3 and its continuity is investigated. The stability problem for PWA systems is presented in Section 4. The numerical examples illustrated in Section 5 confirm the effectiveness of the approach. Section 6 concludes the paper.

2. Preliminaries

Let us recall some useful definitions and concepts.

Definition 1. Given a finite number ρ of points $\{r_\ell\}_{\ell=1}^{\rho}$, $r_\ell \in \mathbb{R}^n$, $\rho \in \mathbb{N}$, a conical hull $\mathcal{C} = cone\,\{r_\ell\}_{\ell=1}^{\rho}$ is the set of points $v \in \mathbb{R}^n$ such that $v = \sum_{\ell=1}^{\rho} \theta_\ell r_\ell$, with $\theta_\ell \in \mathbb{R}_+$, $\mathbb{R}_+$ being the set of nonnegative real numbers. The set $\mathcal{C}$ is also called (polyhedral) cone and the points $\{r_\ell\}_{\ell=1}^{\rho}$ are called rays of the cone. The matrix $R \in \mathbb{R}^{n\times\rho}$ whose columns are the points $\{r_\ell\}_{\ell=1}^{\rho}$ in an arbitrary order is called ray matrix. Any $v \in \mathcal{C}$ can be written as $v = R\theta$ where $\theta \in \mathbb{R}_+^{\rho}$.

Definition 2. Given a finite number λ of points $\{v_\ell\}_{\ell=1}^{\lambda}$, $v_\ell \in \mathbb{R}^n$, $\lambda \in \mathbb{N}$, a convex hull, say $conv\{v_\ell\}_{\ell=1}^{\lambda}$, is a conical hull with $\sum_{\ell=1}^{\lambda} \theta_\ell = 1$.

Definition 3. Given a finite number λ of vertices $\{v_\ell\}_{\ell=1}^{\lambda}$ and a finite number ρ of rays $\{r_\ell\}_{\ell=1}^{\rho}$, $v_\ell, r_\ell \in \mathbb{R}^n$, $\lambda, \rho \in \mathbb{N}$, the (convex) set

$$X = conv\{v_\ell\}_{\ell=1}^{\lambda} + cone\{r_\ell\}_{\ell=1}^{\rho} \tag{1}$$

is a polyhedron in $\mathbb{R}^n$. The expression (1) identifies the so-called $\mathcal{V}$*-representation* of the polyhedron.

In the following we assume that in the polyhedron representation (1) all possible redundancies of the set of vertices and rays have been eliminated.

Any non-empty polyhedron can be equivalently represented by using the $\mathcal{H}$-representation or the $\mathcal{V}$-representation (Avis, Fukuda, & Picozzi, 2002). Given an $\mathcal{H}$-representation of a polyhedron there exist numerical tools for obtaining a corresponding $\mathcal{V}$-representation, e.g., Fukuda (2016).

Definition 4. Denote by $\mathrm{int}(X)$ the interior of a full-dimensional set $X \subseteq \mathbb{R}^n$ and S a finite positive integer. A *partition* of X is the family of full-dimensional sets $\{X_s\}_{s=1}^{S}$ satisfying $X = \cup_{s=1}^{S} X_s$ and $\mathrm{int}(X_s) \cap \mathrm{int}(X_m) = \emptyset$ for $s \neq m$.

In this paper we are interested in polyhedral partitions of X, i.e., to the case where $\{X_s\}_{s=1}^{S}$ are polyhedra, such that the intersection of two polyhedra is either empty or a common face. If such property does not hold, regions can be subdivided such that the property is fulfilled. An $(n-1)$-dimensional face of a polyhedron is called *facet*.

Given a polyhedron one can define two corresponding cones of interest. The conical hull of a polyhedron X represented as in (1) is the cone $\mathcal{C}_X \subseteq \mathbb{R}^n$ defined as

$$\mathcal{C}_X = cone\{\{v_\ell\}_{\ell=1}^{\lambda},\ \{r_\ell\}_{\ell=1}^{\rho}\}. \tag{2}$$

In the following we assume that (2) is a minimal representation for $\mathcal{C}_X$, in the sense that in (2) all possible redundancies of the set of generators have been eliminated and the numbers λ and ρ redefined accordingly. The matrix $R = (v_1 \ \cdots \ v_\lambda \ \ r_1 \ \cdots \ r_\rho)$, with $R \in \mathbb{R}^{n\times(\lambda+\rho)}$, is the ray matrix of $\mathcal{C}_X$. Note that if $0 \in \mathrm{int}(X)$ then $\mathcal{C}_X$ is equal to $\mathbb{R}^n$. For the analysis of interest in the sequel of the paper it is assumed without loss of generality that if $0 \in X$ then the origin belongs to the boundary of X.

Another cone related to a polyhedron $X \subset \mathbb{R}^n$, denoted by $\hat{\mathcal{C}}_X \subset \mathbb{R}^{n+1}$, is obtained by means of the homogenization procedure defined below.

Definition 5. Consider a polyhedron $X \subset \mathbb{R}^n$ with the representation (1). For each vertex $v_\ell \in \mathbb{R}^n$, its vertex-homogenization $\bar{v}_\ell \in \mathbb{R}^{n+1}$ is defined as $\bar{v}_\ell = \mathrm{col}(v_\ell, 1) \in \mathbb{R}^{n+1}$, where $\mathrm{col}(\cdot)$ indicates a vector obtained by stacking in a unique column the column vectors in its argument. For each ray $r_\ell \in \mathbb{R}^n$ its direction-homogenization $\bar{r}_\ell \in \mathbb{R}^{n+1}$ is defined as $\bar{r}_\ell = \mathrm{col}(r_\ell, 0) \in \mathbb{R}^{n+1}$.

Given a polyhedron $X \subset \mathbb{R}^n$ it is possible to define a corresponding cone $\hat{\mathcal{C}}_X \subset \mathbb{R}^{n+1}$ by moving X to the hyperplane $\mathcal{H} = \left\{\bar{x} \in \mathbb{R}^{n+1} : \bar{x} = \mathrm{col}(x, 1), x \in \mathbb{R}^n\right\}$ and drawing all the halflines from the origin of $\mathbb{R}^{n+1}$ to any point of X, as stated in the following proposition.

Proposition 6. *Given a polyhedron $X \subset \mathbb{R}^n$, consider the points $\{\bar{v}_\ell\}_{\ell=1}^{\lambda}$ and $\{\bar{r}_\ell\}_{\ell=1}^{\rho}$ in $\mathbb{R}^{n+1}$ obtained by applying the homogenization in Definition 5. Then the cone in $\mathbb{R}^{n+1}$*

$$\hat{\mathcal{C}}_X = cone\{\{\bar{v}_\ell\}_{\ell=1}^{\lambda}, \{\bar{r}_\ell\}_{\ell=1}^{\rho}\} \tag{3}$$

is such that $\hat{\mathcal{C}}_X \cap \mathcal{H} = \bar{X}$ where $\mathcal{H}$ is the hyperplane defined above and $\bar{X} = \left\{\bar{x} \in \mathbb{R}^{n+1} : \bar{x} = \mathrm{col}(x, 1), x \in X\right\}$.

For any cone $\hat{\mathcal{C}}_X \subset \mathbb{R}^{n+1}$ defined by Proposition 6, one can obtain a corresponding ray matrix $\hat{R} \in \mathbb{R}^{(n+1)\times(\lambda+\rho)}$ which has the form

$$\hat{R} = \begin{pmatrix} v_1 & \cdots & v_\lambda & r_1 & \cdots & r_\rho \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}. \tag{4}$$

A nonempty intersection of two polyhedra is a polyhedron. In the stability analysis we will need to formulate the continuity condition of a candidate Lyapunov function over the polyhedra intersections. To this aim we will exploit the following result.

Lemma 7. *Given two polyhedra $X_1, X_2 \subset \mathbb{R}^n$ such that $X_1 \cap X_2 \neq \emptyset$, then $\hat{\mathcal{C}}_{X_1\cap X_2} = \hat{\mathcal{C}}_{X_1} \cap \hat{\mathcal{C}}_{X_2}$.*

Proof. The proof easily follows by applying the homogenization procedure and then the definitions of polyhedron and cone.

We can now present some definitions and results on copositivity and cone-copositivity.

Definition 8. A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is cone-copositive with respect to a cone $\mathcal{C} \subseteq \mathbb{R}^n$ if it is positive semidefinite with respect to that cone, i.e., if $x^\top P x \geq 0$ for any $x \in \mathcal{C}$. A cone-copositive matrix will be denoted by $P \succcurlyeq^{\mathcal{C}} 0$. If the equality only holds for $x = 0$, then P is strictly cone-copositive and the notation is $P \succ^{\mathcal{C}} 0$. In the particular case $\mathcal{C} = \mathbb{R}_+^n$, a (strictly) cone-copositive matrix is called (strictly) copositive.

The notation $P \succcurlyeq 0$, i.e., without any superscript on the inequality, indicates that P is positive semidefinite, i.e., $x^\top P x \geq 0$ for any $x \in \mathbb{R}^n$. The cone-copositivity evaluation of a known symmetric matrix P on a cone can be always transformed into an equivalent copositive problem and then to an LMI, as stated by the following result.