ARTICLE IN PRESS
Automatica ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

Stabilization for the multi-dimensional heat equation with disturbance on the controller☆

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ARTICLE INFO

Article history:
Received 29 March 2016
Received in revised form
15 January 2017
Accepted 7 March 2017
Available online xxxx

Keywords:
Heat equations
Feedback
Active disturbance rejection control

ABSTRACT

In this paper, we consider stabilization for a multi-dimensional heat equation with internal control matched disturbance. The active disturbance rejection control approach is adopted in this work. First of all, we estimate the disturbance in terms of the output. Then we cancel the disturbance by its approximation. In the end, we design some control strategies to stabilize the anti-stable system.

1. Introduction

In this work, we consider the stabilization of the following multi-dimensional heat equation:

$$\begin{cases} y_t(x,t) - \triangle y(x,t) - ay(x,t) = \chi_\omega(U(x,t) + d(x,t)), \\ (x,t) \in \Omega \times (0,+\infty), \\ y(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,+\infty), \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^d$ $(d \geq 1)$ with Lipschitz boundary, $\omega \subset \Omega$ is an nonempty subdomain of Ω, χ_ω is the characteristic function of ω, $a > 0$ is a fixed number, $U(x,t)$ is the control function, and $d(x,t)$ can be regarded as an uncertainty disturbance on the control, which comes from outside of the system. Usually, the disturbance on the control cannot be avoided when control plans are carried out in the system. In this paper, we suppose that

$$\begin{cases} \chi_\omega d(x,t) \in L^\infty(0,\infty; L^2(\omega)), \\ \chi_\omega d_t(x,t) \in L^\infty(0,\infty; L^2(\omega)). \end{cases} \tag{2}$$

The heat equation is one of the most important second order partial differential equations, and this controlled system plays an important role in industry application for temperature control (see Christofides, 1998). It is well known that the Dirichlet boundary value problem

$$\begin{cases} y_t(x,t) - \triangle y(x,t) - ay(x,t) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ y(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,+\infty). \end{cases} \tag{3}$$

is unstable if a is bigger than the first eigenvalue of $-\triangle$. In the case without disturbance, system (1) can be exponentially stabilized with the feedback control

$$U(x,t) = -ky(x,t), \quad (x,t) \in \Omega \times (0,+\infty), \tag{4}$$

where k is a positive number big enough (see Barbu, 2013; Barbu, Lefter, & Tessitore, 2002 and Barbu & Wang, 2003). However, control (4) fails to deal with the disturbance on the control. To the best of our knowledge, this problem for PDEs was first studied in Guo and Jin (2013) for the stability of the one-dimensional anti-stable wave equation. The main idea in Guo and Jin (2013) is that the disturbance can be estimated in terms of the outputs, and then it can be canceled. This strategy is also called the active disturbance rejection control (ADRC) strategy, which was first proposed by Han in Han (2009). The results of the ADRC for general nonlinear lumped parameter systems are available recently in Guo and Zhao (2013). Another important paper we should mention is Krstic and Smyshlyaev (2008). In Krstic and Smyshlyaev (2008), the authors introduced the adaptive design to handle the parabolic PDEs with disturbance and anti-damping. For other related works on this subject, we refer to Freidovich and Khalil (2008), Guo and Guo (2013b), Han (2009), Krstic (2010), Krstic, Guo, Balogh, and Smyshlyaev (2008), Vazqueza and Krstic (2008) and Smyshlyaev and Krstic (2007a,b), and the references therein. It should be pointed out that many control methods aforementioned have also been applied to

☆ This work was partially supported by the Natural Science Foundation of Henan Province (No. 162300410176). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Xiaobo Tan under the direction of Editor André L. Tits.

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http://dx.doi.org/10.1016/j.automatica.2017.04.011
0005-1098/

deal with uncertainties in PDEs. However, there is few work on stabilization for multi-dimensional PDEs. In our work, we mainly deal with the stabilization of multi-dimensional anti-stable heat equations with a disturbance on the control.

The paper is organized as follows. In Section 2, we will present the main result and its proof. The numerical simulation will be given in Section 3.

2. The main result

In this work, we study the stability of system (1) when (2) holds. The objective of our work is to design a continuous control $U(x,t)$, which is based on the output, to stabilize system (1) with disturbance $d(x,t)$ on the control.

First of all, we introduce some notations. Let $\{\lambda_i\}_{i=1}^{\infty}$, $0<\lambda_1<\lambda_2\le\cdots\le\lambda_n\ldots$, be the eigenvalues of $-\triangle$ with the Dirichlet boundary condition on Ω, and $\{e_i\}_{i=1}^{\infty}$ be the corresponding eigenfunctions satisfying that $\|e_i(x)\|_{L^2(\Omega)}=1$, $i=1,2,3\ldots$, which constitutes an orthonormal basis for $L^2(\Omega)$. Then, the disturbance function $\chi_\omega d(x,t)$ can be written as

$$\chi_\omega d(x,t)=\sum_{i=1}^{\infty}d_i(t)e_i(x), \tag{5}$$

where $d_i(t)=\int_\Omega e_i(x)\chi_\omega d(x,t)dx$, $i=1,2,3\ldots$, are the Fourier coefficients of $\chi_\omega d(x,t)$. In addition to (2), we suppose that the following condition holds.

- Condition (C): The Fourier series (5) is uniformly convergent in $L^2(\Omega)$ for any $t\in[0,+\infty)$. Namely, for any $\delta>0$, there exists a positive number $N=N(\delta)$, which is independent of t, such that

$$\left\|\sum_{i>N}d_i(t)e_i(x)\right\|_{L^2(\Omega)}<\delta,\quad \text{for any } t\in[0,+\infty). \tag{6}$$

Remark 2.1. (1) In Guo and Guo (2013a,b), and Guo, Guo, and Shao (2011), the authors discussed the case that the disturbance function was taken as harmonic disturbance, in which case Condition (**C**) holds naturally.
(2) If the disturbance function $d(x,t)=f(t)G(x)$, where $G(x)\in L^2(\Omega)$ and $f(t)$ is a bounded function from $[0,+\infty)$ to $\mathbb{R}$, then Condition (**C**) also holds.

In this paper, we suppose the output measurement to be

$$Y_i(t)=\int_\Omega e_i(x)y(x,t)dx,\quad i=1,2,3\ldots. \tag{7}$$

By direct computation,

$$\begin{aligned}Y_i'(t)&=\int_\Omega e_i(x)y_t(x,t)dx\\&=-\lambda_iY_i(t)+aY_i(t)+\int_\Omega e_i(x)\chi_\omega U(x,t)dx+d_i(t).\end{aligned} \tag{8}$$

We design a state observer to estimate $Y_i(t)$ and $d_i(t)$ as in Guo and Zhao (2011):

$$\begin{cases}\hat{Y}_{i\varepsilon}'(t)=\left(\int_\Omega e_i(x)\chi_\omega U(x,t)dx+\widehat{d}_{i\varepsilon}(t)\right)\\ \quad+(a-\lambda_i)Y_i(t)+\dfrac{1}{\varepsilon}\left(Y_i(t)-\hat{Y}_{i\varepsilon}(t)\right),\\ \hat{d}_{i\varepsilon}'(t)=\dfrac{1}{4\mu\varepsilon^2}\left(Y_i(t)-\hat{Y}_{i\varepsilon}(t)\right)\end{cases} \tag{9}$$

where $\varepsilon>0$ is the small tuning parameter, then the errors

$$\tilde{Y}_{i\varepsilon}(t)=Y_i(t)-\hat{Y}_{i\varepsilon}(t),\qquad \tilde{d}_{i\varepsilon}(t)=d_i(t)-\hat{d}_{i\varepsilon}(t) \tag{10}$$

satisfy

$$\begin{cases}\tilde{Y}_{i\varepsilon}'(t)=\tilde{d}_{i\varepsilon}(t)-\dfrac{1}{\varepsilon}\tilde{Y}_{i\varepsilon}(t),\\ \tilde{d}_{i\varepsilon}'(t)=-\dfrac{1}{4\mu\varepsilon^2}\tilde{Y}_{i\varepsilon}(t)+d_i'(t).\end{cases} \tag{11}$$

Lemma 2.1. *Suppose that (2) holds, and $\mu>1$. Then, for any given $T>0$, it follows that*

$$|\tilde{Y}_{i\varepsilon}(t)|+|\tilde{d}_{i\varepsilon}(t)|\to 0, \tag{12}$$

as $\varepsilon\to 0$, uniformly for $t\in[T,+\infty)$, and $i=1,2,3\ldots$.

Remark 2.2. While the main idea is derived from Theorem 2.1 of Guo and Zhao (2011) for the case of ODEs, we would rather give the proof here in detail for the sake of completeness.

Proof. Let $\tilde{Z}_{i\varepsilon}(t)=\frac{1}{\varepsilon}\tilde{Y}_{i\varepsilon}(t)$, $i=1,2,3\ldots$, in (11),

$$\Phi_{i\varepsilon}(t)=\begin{pmatrix}\tilde{Z}_{i\varepsilon}(t)\\ \tilde{d}_{i\varepsilon}(t)\end{pmatrix},\qquad D_i(t)=\begin{pmatrix}0\\ d_i'(t)\end{pmatrix},$$

$$A=\begin{pmatrix}-1&1\\ -\dfrac{1}{4\mu}&0\end{pmatrix}.$$

Then, we can rewrite (11) as

$$\Phi_{i\varepsilon}'(t)=\frac{1}{\varepsilon}A\Phi_{i\varepsilon}(t)+D_i(t). \tag{13}$$

The eigenvalues of A are

$$\mu_1=-\frac{1}{2}\left(1-\sqrt{1-\frac{1}{\mu}}\right),\qquad \mu_2=-\frac{1}{2}\left(1+\sqrt{1-\frac{1}{\mu}}\right).$$

It is easy to check that $\mu_2<\mu_1<0$, when $\mu>1$. Thus, there exists a constant $L>0$, which is independent of ε, such that $\|e^{\frac{1}{\varepsilon}At}\|\le Le^{\frac{1}{\varepsilon}\mu_1t}$. The solution of (13) can be written as

$$\Phi_{i\varepsilon}(t)=e^{\frac{1}{\varepsilon}At}\Phi_{i\varepsilon}(0)+\int_0^t e^{\frac{1}{\varepsilon}A(t-\tau)}D_i(\tau)d\tau. \tag{14}$$

It follows that

$$\begin{aligned}\|\Phi_{i\varepsilon}(t)\|_{\mathbb{R}^2}&\le\left\|e^{\frac{1}{\varepsilon}At}\Phi_{i\varepsilon}(0)\right\|_{\mathbb{R}^2}+\left\|\int_0^t e^{\frac{1}{\varepsilon}A(t-\tau)}D_i(\tau)d\tau\right\|_{\mathbb{R}^2}\\&\le Le^{\frac{1}{\varepsilon}\mu_1t}\|\Phi_{i\varepsilon}(0)\|_{\mathbb{R}^2}+\int_0^t Le^{\frac{1}{\varepsilon}\mu_1(t-\tau)}\|D_i(\tau)\|_{\mathbb{R}^2}d\tau\\&\le Le^{\frac{1}{\varepsilon}\mu_1t}\|\Phi_{i\varepsilon}(0)\|_{\mathbb{R}^2}+\frac{L\varepsilon}{\mu_1}\|d_i'(t)\|_{L^\infty(0,\infty)}.\end{aligned} \tag{15}$$

By (2) and (5), we obtain that $\|d_i'(t)\|_{L^\infty(0,\infty)}$ are uniformly bounded for $i=1,2,3\ldots$. Thus, for any $T>0$,

$$\left|\frac{1}{\varepsilon}\tilde{Y}_{i\varepsilon}(t)\right|+|\tilde{d}_{i\varepsilon}(t)|\to 0,\quad \text{as } \varepsilon\to 0, \tag{16}$$

uniformly for $t\in[T,+\infty)$, and $i=1,2,3\ldots$. This shows (12) holds. □

Please cite this article in press as: Zheng, G., & Li, J., Stabilization for the multi-dimensional heat equation with disturbance on the controller. Automatica (2017), http://dx.doi.org/10.1016/j.automatica.2017.04.011