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Brief paper

Region of attraction estimation using invariant sets and rational Lyapunov functions☆

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ABSTRACT

This work addresses the problem of estimating the region of attraction (RA) of equilibrium points of nonlinear dynamical systems. The estimates we provide are given by positively invariant sets which are not necessarily defined by level sets of a Lyapunov function. Moreover, we present conditions for the existence of Lyapunov functions linked to the positively invariant set formulation we propose. Connections to fundamental results on estimates of the RA are presented and support the search of Lyapunov functions of a rational nature. We then restrict our attention to systems governed by polynomial vector fields and provide an algorithm that is guaranteed to enlarge the estimate of the RA at each iteration.

1. Introduction

The problem of computing the region of attraction (RA) of asymptotically stable equilibria, or inner estimates to this set (ERA) (Chiang, Hirsch, & Wu, 1988), is central in several applications and its relevance is immediately clear for many practical nonlinear systems for which we can only guarantee local properties of operating points.

With a converse Lyapunov theorem (Zubov, 1964, Theorem 19), Zubov answered the question "*[...] Is it possible, with the help of the Lyapunov function to find a region of variation of the initial values* x_0 *such that* $\|\phi(t, x_0)\| \to 0$ as $t \to \infty$?" (Zubov, 1964, p. 3). The theorem states that if $\mathcal{S}$ *is* the RA of an equilibrium then the existence of a Lyapunov function (LF) satisfying some conditions on such a set $\mathcal{S}$ is *necessary and sufficient*. However, computing the LF and the exact RA following Zubov's theorem requires the solution of a partial differential equation, which is difficult to obtain in all but simple cases. However, *local* solutions (in a compact set around the equilibrium point) can be obtained more easily and yield ERAs for the equilibrium point of interest. In this context, a method to approximate solutions to the conditions of Zubov (1964, Theorem 19) is obtained with a series expansion of the LF (Zubov, 1964, p. 91) and is now referred to as *Zubov's Method*.

In Vannelli and Vidyasagar (1985), Zubov's theorem was modified to consider Lyapunov functions mapping $\mathbb{R}^n$ to $\mathbb{R}_{\geq 0}$ (the original result is stated in terms of a map from $\mathbb{R}^n$ to the interval $[-1, 0]$). One of the conditions in Vannelli and Vidyasagar (1985) imposes that the LF $V(x)$ satisfies $V(x) \to \infty$ whenever $x \to \partial\mathcal{S}$ (the boundary of the RA) or whenever $\|x\| \to \infty$. Such a property is described by the observation that *"the candidate must in effect 'blow up' near the boundary of the domain of attraction"*. These functions were called *maximal Lyapunov functions* (MLFs). One of the key observations was that rational functions could be used to approximate MLFs and therefore be used to obtain estimates of the RA. As a matter of fact, the class of rational functions of the form $V(x) = \frac{V_N(x)}{V_D(x)}$ where V_N and V_D are polynomials, was considered as LF candidates in the algorithm proposed in Vannelli and Vidyasagar (1985) with the boundary of the ERA characterised by the set $\{x \in \mathbb{R}^n \mid V_D(x) = 0\}$.

For the sake of clarity, it is important to distinguish between two similar sounding yet very different objects: a maximal Lyapunov function (MLF) and a maximal Lyapunov set (MLS). An MLF is a Lyapunov function which satisfies a strict set of conditions (cf. Definition 1 in Section 3). In contrast, a MLS is defined as the largest level set of a *given* LF contained in a specified set. Computing the MLS is of interest since one might wish to compute the best

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ERA achievable for a given Lyapunov function (Chesi, 2004, 2013). Further to the choice of the class of the LF, conservativeness is introduced by imposing the level sets of the Lyapunov function to be the ERA, as observed in Khalil (2002, p. 320) *"Estimating the region of attraction by $\Omega_c = \{x|V(x) \leq c\}$ is simple but usually conservative. According to LaSalles's theorem [...] we can work with any compact set $\Omega \subset D$ provided we can show that Ω is positively invariant"*. The statement highlights the fact that contractiveness of the function defining the ERA is restrictive.

In recent years, sufficient conditions for local stability analysis, requiring invariance and contractiveness of a set led to numerical methods for the estimation of the RA with polynomial Lyapunov functions (Tan & Packard, 2008; Topcu & Packard, 2009; Topcu, Packard, Seiler, & Balas, 2010). These methods rely on the solution of non-convex sum-of-squares (SOS) constraints constructed with the *Positivstellensatz* (Lasserre, 2009, Theorem 2.14). The solutions to these problems require a coordinate-wise search since the non-convex nature results from the fact that some polynomial variables appear multiplying the Lyapunov function which is itself a variable. For a detailed description of sum-of-squares methods for RA estimation the reader is referred to Chesi (2011). For the case of a given LF, the computation of the MLS was pursued in Chesi (2013). In Henrion and Korda (2014) the theory of moments is used to estimate the RA of polynomial systems. We also find in the literature numerical methods exploiting topological properties of the boundary of the RA requiring the computation of trajectories and equilibrium points. However the complexity of such methods has restricted them to 2-dimensional examples (Chiang et al., 1988). Recently, in Wang, Lall, and Chiou (2011), set advection methods are described for polynomial systems.

In this paper we derive conditions based on Lyapunov stability results that guarantee that trajectories initiated from a *positively invariant set* converge to a level set of some LF which is contractive and invariant therefore guaranteeing such a positively invariant set to be an ERA. In addition to the positively invariant estimates, we present conditions to obtain LF certificates of a specific form which specialises to rational functions in case of polynomial data. We then propose a numerical method based on the solution of SOS constraints for the case of polynomial systems and estimates in the form of semi-algebraic sets (sets defined by polynomial constraints). The work in this paper extends the work of Valmorbida and Anderson (2014) and connects the concept of *maximal Lyapunov functions* (Vannelli & Vidyasagar, 1985) to polynomial optimisation techniques based on sum-of-squares programming. To the best of the authors knowledge this is the first work to offer a theoretical link between maximal Lyapunov functions, which completely characterise the ERA (and can be approximated to arbitrary accuracy by rational functions) and sum-of-squares methods for rational LF construction. Note that rational Lyapunov functions were considered in Chesi (2013) to obtain MLSs.

The paper is organised as follows. The problem is formulated in Section 2 and we present the main theoretical results in Section 3. Narrowing our attention to systems described by polynomial vector fields we describe a computational method for constructing ERAs based on sum-of-squares programming in Section 4 which is illustrated by numerical examples in Section 5.

2. Preliminaries

Let $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{R}_{>0}$ and $\mathbb{R}^n$ denote the sets of real numbers, non-negative real numbers, positive real numbers and the n-dimensional Euclidean space respectively. The function $f : \mathbb{R}^n \to \mathbb{R}$ is positive definite if $f(x) > 0$ for all non-zero $x \in \mathbb{R}^n$, similarly if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$ then f is positive semidefinite. The set of functions $g : \mathbb{R}^n \to \mathbb{R}$ which is n-times continuously differentiable is denoted $\mathcal{C}^n$. $co(\mathcal{X})$ denotes the convex hull of the set $\mathcal{X}$, $\mathcal{X}^\circ$ its interior, $\partial\mathcal{X}$ its boundary, and $\overline{\mathcal{X}}$ its closure. The minimum (maximum) of a scalar function $S(x)$ in a compact set $\mathcal{Y}$ is denoted $\min_{x\in\mathcal{Y}}(S(x))$ $(\max_{x\in\mathcal{Y}}(S(x)))$. We also use max to denote the function taking the maximum of its arguments. For $x \in \mathbb{R}^m$ the ring of polynomials in m variables is denoted by $\mathbb{R}[x]$. For $p \in \mathbb{R}[x]$, $deg(p)$ denotes the degree of p. A polynomial $p(x)$ is said to be a sum-of-squares if there exists a finite set of polynomials $g_1(x), \ldots, g_k(x)$ such that $p(x) = \sum_{i=1}^k g_i^2(x)$. The set of SOS polynomials in x is denoted by $\Sigma[x_1, \ldots, x_m]$ which can be abbreviated to $\Sigma[x]$. Equivalently, $p(x)$ is SOS if there exists a positive semidefinite matrix Q such that $p(x) = Z^T(x)QZ(x)$ where $Z(x)$ is a vector of monomials (Parrilo, 2000). Note that the search for Q can be formulated as a semidefinite programme and thus solved using convex optimisation techniques (Vandenberghe & Boyd, 1996).

Consider the dynamical system

$$\dot{x} = f(x) \tag{1}$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $\mathcal{D} \subset \mathbb{R}^n$ to $\mathbb{R}^n$, with $0 \in \mathcal{D}$. Let us assume $x = 0$ is an equilibrium point, *i.e.* $0 \in \{x \in \mathbb{R}^n | f(x) = 0\}$. Denote by $\phi(t, x(0))$ the solution to (1) that is initiated from the point $x(0)$ at time $t = 0$, the set L is said to be *invariant* with respect to (1) provided $x(0) = \phi(0, x(0)) \in L \Rightarrow x(t) = \phi(t, x(0)) \in L, \forall t \in \mathbb{R}$. Furthermore, L is said to be *positively invariant* with respect to (1) if the previous implication holds for all $t \geq 0$. Given a (continuous) function $R : \mathbb{R}^n \to \mathbb{R}$ we define the set $\mathcal{E}(R, \gamma) := \{x \in \mathbb{R}^n | R(x) \leq \gamma\}$ for some $\gamma > 0$ and so $\mathcal{E}^\circ(R, \gamma) = \{x \in \mathbb{R}^n | R(x) < \gamma\}$ for the same γ. Additionally, provided that a function $V : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}_{>0}$ satisfies $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0$ on $\mathcal{E}(V, \gamma)$ then the set $\mathcal{E}(V, \gamma)$ is said to be *contractive* and *invariant*, furthermore the function V is said to be a Lyapunov function (Khalil, 2002, Chapter 4). We assume throughout this work that any function used to define a contractive set is in class $\mathcal{C}^1$. The region of attraction of an asymptotically stable equilibrium point x^* of (1) is defined as the set

$$\mathcal{S} := \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \phi(t, x) \text{is defined } \forall\, t \geq 0, \\ \lim\limits_{t\to\infty} \phi(t, x) = x^* \end{array} \right\}, \tag{2}$$

without loss of generality, we will assume throughout this paper that the equilibrium point of interest is at the origin, i.e. $x^* = 0$.

The focus of this paper is to construct inner estimates of $\mathcal{S}$ by computing positively invariant sets.

3. Main results

In this section we present conditions to certify that a compact set is a positively invariant set and provides an estimate of the RA for the origin of (1). Furthermore, it is shown how, under an additional condition the estimate of the RA is characterised by a Lyapunov function. We then extend these results to the case where the system under study is affected by parametric uncertainty.

3.1. Region of attraction estimates

The following theorem verifies that a compact set is positively invariant and defines an estimate of the RA of the equilibrium point at the origin, and allows us to obtain functions of which the denominator provides the RA estimate.

Theorem 1. *Given $R : \mathbb{R}^n \to \mathbb{R}$, $R \in \mathcal{C}^1$ and $\gamma > 0$, satisfying*

$$\mathcal{E}(R, \gamma) \subset \mathcal{D} \text{ is compact and } 0 \in \mathcal{E}(R, \gamma), \tag{3a}$$

$$-\langle \nabla R(x), f(x) \rangle > 0 \quad \forall\, x \in \partial\mathcal{E}(R, \gamma), \tag{3b}$$