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Existence of an observation window of finite width for continuous-time autonomous nonlinear systems☆

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ABSTRACT

In this note, the relationship between notions of observability for continuous-time nonlinear system related to distinguishability, observability rank condition and K-function has been investigated. It is proved that an autonomous nonlinear system that is observable in both distinguishability and rank condition sense permits an observation window of finite width, and it is possible to construct a K-function related to observability for such system.

1. Introduction

The state estimation problem is one of the most fundamental problems in control system theory. The intrinsic property of the system that makes the state estimation problem feasible is observability, and this property has been extensively studied in past decades.

For linear systems, the notion of observability is firmly established (Kailath, 1980). Contrary, for nonlinear systems, several non-equivalent definitions of observability have been proposed, and by using these definitions, the state estimation problem of nonlinear systems have been extensively studied (Alamir, 2007; Besançon, 2007; Gauthier, Hammouri, & Othman, 1992; Hammouri, 2007; Hermann & Krener, 1977; Isidori, 1995; Nijmeijer & van der Schaft, 1990). However, the relations between several different notions of observability have not been fully understood (although there are several established facts Besançon, 2007; Hammouri, 2007; Hermann & Krener, 1977).

Recently, the author has proved that a discrete-time nonlinear systems whose observation map is injective and the Jacobian of the observation map is of full rank satisfies a seemingly stronger condition called uniform observability (Hanba, 2009), and by assuming these conditions, it is possible to construct a $\mathcal{K}$-function related to observability (Hanba, 2010). This note is an attempt to establish corresponding results for continuous-time systems.

The scope of this note is limited to autonomous nonlinear systems, and we deal with three typical definitions of observability for nonlinear systems which are related to distinguishability, rank condition and $\mathcal{K}$-function, respectively (precise definitions are given later). Roughly speaking, we prove that distinguishability together with the observability rank condition implies that there is a 'observation window' (the sequence of past output as a function of time) of finite width which determines the initial state uniquely, and it is possible to construct a $\mathcal{K}$-function related to observability.

A preliminary version of this manuscript is available in arXiv (Hanba, 2015).

2. Main results

In this note, we consider an autonomous nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x), \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $y \in \mathbb{R}^p$ is the output. The functions $f(x)$ and $h(x)$ are assumed to be of compatible dimensions, and smooth up to required order. The solution of (1) is assumed to be unique, and the solution initialized at $t = 0$ by x_0 is denoted by $\varphi(t, 0, x_0)$, which is assumed to be a continuous function of x_0.

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The set of permissible initial conditions of (1) is assumed to be compact, and is denoted by Ω. Note that the state space itself is not necessarily compact.

Next, we introduce the definitions of observability considered in this note. Unfortunately, there is no general agreement on how to name these properties. Hence, we temporally call them 'D-observability', 'R-observability', and 'K-observability' (to be defined below) for brevity.

Definition 1 (*Besançon, 2007*)**.** A pair of initial states (x_1, x_2) of (1) with $x_1 \neq x_2$ is said to be an indistinguishable pair if $\forall t \geq 0$, $h(\varphi(t, 0, x_1)) = h(\varphi(t, 0, x_2))$.

Definition 2 (*Besançon, 2007*)**.** The system (1) is said to be D-observable (with respect to Ω) if there is no indistinguishable pair in the set Ω.

Definition 3 (*Besançon, 2007*)**.** The system (1) is said to be R-observable (with respect to Ω) if $\exists N > 0$, the Jacobian of the map $H(x) = (h(x), L_f h(x), \ldots, L_f^{N-1} h(x))$ is of full rank on Ω, where $L_f h = \frac{\partial h}{\partial x} f$, and $L_f^k h = L_f \left(L_f^{k-1} h\right)$.

Definition 4 (*Haddad & Chellaboina, 2008; Khalil, 1996*)**.** A function $\alpha : D \to [0, \infty)$ (where D is either $[0, \infty)$, $[0, a)$ or $[0, a]$ with $a > 0$) is said to be a $\mathcal{K}$-function if it is continuous, $\alpha(0) = 0$, and is strictly increasing.

Definition 5 (*Alamir, 2007*)**.** The system (1) is said to be K-observable (with respect to Ω) if $\exists T > 0$, $\forall x_1, x_2 \in \Omega$,

$$\int_0^T |h(\varphi(t, 0, x_1)) - h(\varphi(t, 0, x_2))|^2 \, dt \geq \alpha(|x_1 - x_2|), \tag{2}$$

where $\alpha(\cdot)$ is a $\mathcal{K}$-function and $|\cdot|$ denotes the Euclid norm of a vector.

Remark 1. Each of above definitions require smoothness of $f(x)$ and $h(x)$ in different level.

- For Definition 2, the only requirement is that the system (1) has a unique solution. A finite escape time is allowed, as far as the state distinction is achievable before the arrival of the finite escape time. There is no restriction to $h(x)$.
- For Definition 3, $h(x)$ should be $N - 1$ times continuously differentiable, and $f(x)$ should be $N - 2$ times continuously differentiable, but the value of N cannot be specified (although it is finite).
- For Definition 5, the requirements are that (1) has a unique solution, the solution of (1) is defined for $t \in [0, T]$, and $h(\varphi(t, 0, x_0))$ is integrable for each $x_0 \in \Omega$.

It is a known fact that, if a system is R-observable at a point x_0, then it is D-observable on a neighborhood of x_0 (Besançon, 2007; Hermann & Krener, 1977), but it is not always possible to extend the result to the whole of Ω. On the other hand, D-observability does not imply R-observability, as the following example shows.

Example 1. Consider a 1-dimensional system

$$\begin{aligned} \dot{x} &= -x, \\ y &= h(x) = x^3. \end{aligned} \tag{3}$$

This system is D-observable because it is possible to directly calculate x from y ($x = y^{1/3}$), but is not R-observable at $x = 0$, because $h(x) = x^3$, $L_f h(x) = -3x^3$, and inductively, $L_f^k h(x) = (-1)^k 3^k x^3$, and hence their derivatives vanish at $x = 0$.

It is desirable that the width of the 'observation window' (the time interval that the output of the system is stored in order to determine the initial state uniquely) is finite. In this sense, K-observability is convenient, and has been widely adopted in works on moving horizon state estimation (Alamir, 2007; Alessandri, Baglietto, & Battistelli, 2008). If the system (1) is K-observable, then for $x_1, x_2 \in \Omega$ with $x_1 \neq x_2$, $\exists t : 0 \leq t \leq T$, $h(\varphi(t, 0, x_1)) \neq h(\varphi(t, 0, x_2))$, hence (1) is D-observable. Then, a natural question arises: do systems that are D-observable always permit an observation window of finite width? Unfortunately, the answer is negative, which is given in the following example.

Example 2. Consider a 1-dimensional system

$$\begin{aligned} \dot{x} &= x \\ y &= h(x) = \begin{cases} 0 & x < M \\ x - M & x \geq M, \end{cases} \end{aligned} \tag{4}$$

where M is a positive constant. If the initial condition is zero, then the output is identically zero. For an initial condition $x_0 > 0, x(t) = \exp[t]x_0$, and hence the output is identically zero for $t < \ln(M/x_0)$ and is $\exp[t]x_0 - M$ for $t \geq \ln(M/x_0)$. Hence, the zero initial condition and x_0 cannot be distinguished until $t = \ln(M/x_0)$, and hence as the initial condition gets smaller, the required width of the observation window tends to infinity.

One may argue that the reason for making the width of the observation window infinite is the non-differentiability of the output function, but this is not the case. For example, by replacing the output function $h(x)$ of (4) with

$$h(x) = \begin{cases} 0 & x \leq M \\ \exp[-1/(x - M)] & x \geq M, \end{cases}$$

a similar conclusion holds.

Thus far, we have seen that there are gaps between D-observability, R-observability and K-observability, and a D-observable system does not always permit an observation window of finite width. In the following, we show that, if (1) is D-observable as well as R-observable, then there exists an observation window of finite width, and it is possible to construct a $\mathcal{K}$-function corresponding to Definition 5, and hence (1) is K-observable.

Proposition 1. *If (1) is D-observable as well as R-observable for the initial condition set Ω, then there is a finite $T > 0$ such that $\forall x_1, x_2 \in \Omega$ with $x_1 \neq x_2$, $\exists t : 0 \leq t \leq T$, $h(\varphi(t, 0, x_1)) \neq h(\varphi(t, 0, x_2))$.*

Proof. We first prove that

$$\begin{aligned} &\forall x \in \Omega,\ \exists \mathcal{N}(x),\ \forall z_1, z_2 \in \mathcal{N}(x) \text{ with } z_1 \neq z_2, \\ &\forall T > 0,\ \exists t : 0 \leq t \leq T,\ h(\varphi(t, 0, z_1)) \neq h(\varphi(t, 0, z_2)) \end{aligned} \tag{5}$$

by contradiction, where $\mathcal{N}(x)$ denotes an open neighborhood of x. Suppose that (5) is false, that is,

$$\begin{aligned} &\exists x \in \Omega,\ \forall \mathcal{N}(x),\ \exists z_1, z_2 \in \mathcal{N}(x) \text{ with } z_1 \neq z_2, \\ &\exists T > 0,\ \forall t : 0 \leq t \leq T,\ h(\varphi(t, 0, z_1)) = h(\varphi(t, 0, z_2)). \end{aligned} \tag{6}$$

Then, $h(\varphi(t, 0, z_1)) - h(\varphi(t, 0, z_2))$ is identically zero as a function of t. Hence, for all $k \geq 0$, $\frac{d^k}{dt^k} h(\varphi(t, 0, z_1)) = \frac{d^k}{dt^k} h(\varphi(t, 0, z_2))$, hence $H(\varphi(t, 0, z_1)) = H(\varphi(t, 0, z_2))$, and by letting $t = 0$, $H(z_1) = H(z_2)$. On the other hand, because the Jacobian of H is of full rank, there is a neighborhood of x in which H is injective. By choosing such neighborhood $\mathcal{N}(x)$ (recall that $\mathcal{N}(x)$ is arbitrary), it follows that $H(z_1) \neq H(z_2)$ because $z_1 \neq z_2$, hence a contradiction has been obtained. Therefore, (6) is false and hence (5) is true.