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On the Newton–Kleinman method for strongly stabilizable infinite-dimensional systems[☆]

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ABSTRACT

We consider the Newton–Kleinman method for strongly stabilizable infinite-dimensional systems. Under certain assumptions, the maximal self-adjoint solution to the associated control algebraic Riccati equation is constructed. The constructed solution is also the maximal solution to the corresponding control algebraic Riccati inequality.

1. Introduction

We consider the control algebraic Riccati equation (CARE)

$$A^*X + XA - XBB^*X + C^*C = 0,$$

which has been widely studied in the area of systems and control for both finite-dimensional and infinite-dimensional systems, together with the corresponding control algebraic inequality (CARI)

$$A^*X + XA - XBB^*X + C^*C \geq 0.$$

There are many approaches to the study of the CARE and we will mention only some of them which are most related to this note. One approach is in the line with Willems (1971), where comparison results and a classification of all solutions of the CARE for finite-dimensional systems have been provided. Moreover, the solvability of the CARI implies the solvability of the CARE for finite-dimensional controllable systems (also in Willems, 1971). Necessary and sufficient conditions for the existence of a maximal/minimal solution of the CARE or CARI for sign-controllable systems can be found in Scherer (1991). A comparison of the solutions of two CARE associated to different systems was provided for the first time in Wimmer (1985). A second approach is in terms of the Hamiltonian (see e.g. Martensson, 1971). A third approach is based on Kleinman (1968), and provides iterative procedures for constructing hermitian solutions of the CARE as proposed in Gohberg, Lancaster, and Rodman (1986) and Ran and Vreugdenhil (1988). Moreover, it has been shown in Gohberg et al. (1986) and Ran and Vreugdenhil (1988) that solvability of the CARI implies the existence of a maximal solution of the CARI provided that (A, B) is stabilizable. Furthermore, the maximal solution of the CARI also satisfies the CARE.

Some of the results mentioned above for finite-dimensional systems have been extended to infinite-dimensional systems $\Sigma(A, B, C)$. Following the approach in Willems (1971), a classification of all non-negative self-adjoint solution of the CARE for exponentially stabilizable systems has been proposed in Callier, Dumortier, and Winkin (1995) and for discrete-time infinite-dimensional systems in Malinen (2000). In Iftime, Curtain, and Zwart (2005) a representation of all self-adjoint solutions to the CARE has been obtained under the following assumptions: A generates a strongly continuous semigroup, output stabilizability, strong detectability and the invertibility of the minimal self-adjoint solution of the filter algebraic Riccati equation. Following the approach in Martensson (1971), the relation between the CARE and the eigenvectors of the Hamiltonian for Riesz spectral systems has been studied in Kuiper and Zwart (1993).

In this note the third approach is followed, known in the literature as the Newton–Kleinman method. The finite-dimensional case has been initiated in Kleinman (1968). Since then, this problem has been widely studied for finite-dimensional case (see for example Ran & Vreugdenhil, 1988 and the references therein) and

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also for infinite-dimensional systems which are exponentially stabilizable (see for example Burns, Sachs, & Zietsman, 2008; Curtain & Rodman, 1990). However, there are many infinite-dimensional systems which are not exponentially stabilizable but have nice stability properties (see for example Oostveen, 2000). We provide a Newton–Kleinman type result for strongly stabilizable infinite-dimensional systems. More precisely, provided that $\Sigma(A, B, C)$ is strongly stabilizable and additional assumptions are satisfied, an iterative procedure for constructing the maximal self-adjoint solution to the CARE as the strong limit of Newton–Kleinman iterates. Moreover, it is shown that the maximal solution of the CARI also satisfies CARE.

This note is structured as follows. In Section 2 the necessary concepts are introduced and two preliminary results (Theorem 2.2 and Lemma 2.5) are stated. The main result is Theorem 3.4 and it is presented in Section 3.

Notation: $\mathcal{L}(U, Z)$ denotes the set of bounded operators from the Hilbert space U to the Hilbert space Z. Let B^* be the adjoint operator of $B \in \mathcal{L}(U, Z)$. For self-adjoint operators $X_1, X_2 \in \mathcal{L}(Z)$, by $X_1 \geq X_2$ we understand $X_1 - X_2 \geq 0$.

2. Preliminaries

Let Z, Y and U be separable Hilbert spaces. Consider a state linear system $\Sigma(A, B, C)$ given by

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_0 \in Z \\ y(t) = Cz(t), \end{cases} \tag{1}$$

where $B \in \mathcal{L}(U, Z)$, $C \in \mathcal{L}(Z, Y)$, $z \in Z$, $y \in Y$ and $u \in U$. The unbounded operator $A : \mathcal{D}(A) \subset Z \to Z$ generates a strongly continuous semigroup $T(t)$. If $T(t)z \to 0$ as $t \to \infty$ (i.e. $\lim_{t\to\infty} \|T(t)z\|_Z = 0$) for all $z \in Z$, then $T(t)$ is called a *strongly stable semigroup*.

Recall now the notions of output stability, output stabilizability and strong stabilizability.

Definition 2.1. The system $\Sigma(A, B, C)$ is *output stable* if there exists a constant $\gamma > 0$ such that

$$\int_0^\infty \|CT(s)z\|_Y^2 ds \leq \gamma \|z\|_Z^2, \quad \text{for all } z \in Z.$$

It is known that (see e.g. Grabowski, 1991) output stability of the system $\Sigma(A, B, C)$ is equivalent to the existence of a nonnegative self-adjoint solution $\Pi \in \mathcal{L}(Z)$ of the Lyapunov operator equation

$$\langle Az_1, \Pi z_2\rangle + \langle \Pi z_1, Az_2\rangle = -\langle Cz_1, Cz_2\rangle, \tag{2}$$

where $z_1, z_2 \in \mathcal{D}(A) \subset Z$. The following result is a dual version of Hansen and Weiss (1997, Theorem 3.1). Note that the notion of output stability for the infinite dimensional systems $\Sigma(A, B, C)$ is the same as requiring (in Hansen & Weiss, 1997) that C is an infinite-time admissible operator for $T(t)$.

Theorem 2.2 (Hansen & Weiss, 1997). *Consider the system $\Sigma(A, B, C)$. The following statements are equivalent:*

- (i) *The system $\Sigma(A, B, C)$ is output stable.*
- (ii) *There exists an operator $Q \in \mathcal{L}(Z)$ such that, for every $z \in Z$*

$$Qz := \int_0^\infty T^*(t)C^*CT(t)zdt. \tag{3}$$

- (iii) *There exist operators $\Pi \in \mathcal{L}(Z)$, $\Pi \geq 0$ which satisfy the Lyapunov operator equation (2).*

Moreover, if any of the above statements holds, then the following statements are true:

- (I) *Q defined in (3) satisfies (2) and it is the smallest nonnegative solution of (2).*
- (II) *For any $z \in Z$, $\lim_{t\to\infty} Q^{\frac{1}{2}}T(t)z = 0$. In particular, if $Q > 0$ then $T(t)$ is strongly stable.*
- (III) *If $T(t)$ is strongly stable, then Q is the unique self-adjoint solution of (2).*

Definition 2.3. The system $\Sigma(A, B, C)$ is *output stabilizable* if there exists an $F \in \mathcal{L}(Z, U)$ such that $\Sigma\left(A + BF, B, \begin{bmatrix} F \\ C \end{bmatrix}\right)$is output stable.

Definition 2.4. The system $\Sigma(A, B, C)$ is *strongly stabilizable* if the following conditions are satisfied

- it is output stabilizable by some $F \in \mathcal{L}(Z, U)$, and
- $A_F := A + BF$ generates a strongly stable semigroup $T_F(t)$.

To the system (1) one associates the classical cost to be minimized on infinite-time

$$J(z_0, u) := \int_0^\infty (\langle Cz(s), Cz(s)\rangle + \langle u(s), Ru(s)\rangle)\, ds$$

where R is a self-adjoint coercive operator in $\mathcal{L}(U)$. For the sake of simplicity of the exposition we shall take $R = I$ in the sequel. The cost $J(z_0, u)$ is closely related to the (control) algebraic Riccati equation (see e.g. Curtain & Zwart, 1995)

$$\langle Az_1, Xz_2\rangle + \langle Xz_1, Az_2\rangle - \langle B^*Xz_1, B^*Xz_2\rangle + \langle Cz_1, Cz_2\rangle = 0 \tag{4}$$

for $z_1, z_2 \in \mathcal{D}(A) \subset Z$. Define $\mathcal{R}(X)$ weakly by $\langle z_1, \mathcal{R}(X)z_2\rangle$ being equal to the left hand side of (5). Then (4) can be written as

$$\langle z_1, \mathcal{R}(X)z_2\rangle = 0, \quad z_1, z_2 \in \mathcal{D}(A) \subset Z. \tag{5}$$

Consider also the control algebraic Riccati inequality (CARI)

$$\langle z, \mathcal{R}(X)z\rangle \geq 0, \quad z \in \mathcal{D}(A) \subset Z. \tag{6}$$

Define the sets of self-adjoint solutions of the CARE (5) and the CARI (6) as

$$\mathcal{S}_{\mathcal{E}} := \left\{X \mid X = X^*,\ \langle z_1, \mathcal{R}(X)z_2\rangle = 0,\ z_1, z_2 \in \mathcal{D}(A)\right\} \tag{7}$$

$$\mathcal{S}_{\mathcal{I}} := \left\{X \mid X = X^*,\ \langle z, \mathcal{R}(X)z\rangle \geq 0,\ z \in \mathcal{D}(A)\right\}. \tag{8}$$

One would also need the following result.

Lemma 2.5. *Consider the system $\Sigma(A, B, C)$ such that A generates a strongly stable continuous semigroup $T(t)$. If $L = L^* \in \mathcal{L}(Z)$ satisfies*

$$\langle Az, Lz\rangle + \langle Lz, Az\rangle \leq 0, \quad z \in \mathcal{D}(A) \subset Z, \tag{9}$$

then $L \geq 0$.

Proof. Let $z \in \mathcal{D}(A) \in Z$, $t \geq 0$ and define

$$f(t) := \langle T(t)z, LT(t)z\rangle,$$

which is continuous differentiable. By taking the derivative with respect to t, then integrating on $[0, \tau]$, letting $\tau \to \infty$ and using strong stability of $T(t)$ one has $\langle z, Lz\rangle \geq 0$ on $\mathcal{D}(A)$. Using the density of $\mathcal{D}(A)$ in Z the proof is completed. ■

3. Main results

Consider a strongly stabilizable system $\Sigma(A, B, C)$. Let $F \in \mathcal{L}(Z, U)$ be such that $A_F := A + BF$ generates a strongly continuous