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Non-asymptotic fractional order differentiator for a class of fractional order linear systems☆

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ABSTRACT

This paper aims at designing a non-asymptotic fractional order differentiator for a class of fractional order linear systems to estimate the Riemann–Liouville fractional derivatives of the output in discrete noisy environment. The adopted method is a recent algebraic method originally introduced by Fliess and Sira-Ramirez. Firstly, the fractional derivative of the output of an arbitrary order is exactly given by a new algebraic formula in continuous noise free case without knowing the initial conditions of the considered system. Secondly, a digital fractional order differentiator is introduced in discrete noisy cases, which can provide robust estimations in finite-time. Then, some error analysis is given, where an error bound useful for the selection of the design parameter is provided. Finally, numerical examples illustrate the accuracy and the robustness of the proposed fractional order differentiator.

1. Introduction

Fractional calculus has a long history and has been becoming very useful in many scientific and engineering fields, including control, flow propagation, signal processing, electrical networks, etc. Bandyopadhyay and Kamal (2015), Concepción, Chen, Vinagre, Xue, and Feliu-Batlle (2010), Diethelm (2010), Kilbas, Srivastava, and Trujillo (2006), Miller and Ross (1993) and Podlubny (1998). For instance, fractional order systems and controllers have been applied to improve performance and robustness properties in control design (Sabatier, Farges, Merveillaut, & Fenetau, 2012; Victor, Malti, Garnier, & Oustaloup, 2013; Yin, Chen, & Zhong, 2014), where the fractional derivatives of the output usually need to be estimated from its discrete noisy observation. Consequently, an interesting research topic concerns with designing digital fractional order differentiators, which should be robust against noises.

Various robust fractional order differentiators have been proposed both in the frequency domain and in the time domain. They can be divided into two classes: model-free fractional order differentiators (Chen, Chen, & Xue, 2011; Liu, Gibaru, Perruquetti, & Laleg-Kirati, 2015; Machado, 2009, 2012) and model-based ones (Liu & Laleg-Kirati, 2015; Liu, Tian, Boutat, & Laleg-Kirati, 2015; Wei, Liu, & Boutat, 2016). The first class is obtained by truncating an analytical expression. Hence, this generates a truncated term error which can produce an amplitude error (in the vertical sense) and a shifted error (in the horizontal sense) (Liu et al., 2015). The second class is obtained from the differential equations of considered signals. They do not introduce any truncated term errors.

Existing fractional order differentiators are usually extensions of integer order differentiators, which generally have one of the following disadvantages:

- they are sensible to noises, such as the well-known Grünwald–Letnikov scheme (Concepción et al., 2010), which is the extension of the finite difference scheme and only efficient in noise-free case;
- they produce a truncated term error in the estimated derivatives, such as the model-free fractional order differentiators proposed in Chen et al. (2011) and Liu et al. (2015), which are the extension of classical polynomial approximation methods;
- they asymmetrically converge, for instance, the fractional order Luenberger-like observer, considered as a model-based

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fractional order differentiator, is devoted to estimating the pseudo-state variables which are the fractional derivatives of the output (Sabatier et al., 2012). Moreover, a fractional order observer cannot estimate the fractional derivatives of an arbitrary order.

Among the existing methods, there is a non-asymptotic algebraic method originally introduced by Fliess and Sira-Ramirez for linear identification (Fliess & Sira-Ramírez, 2003). This method permits to obtain exact algebraic integral formulae for the desired estimators, which can provide estimations in finite-time. It has been shown in Fliess (2006) that, thanks to the integral formulae, these estimators exhibit good robustness properties with respect to corrupting noises. By considering these advantages, this method has been extended to design model-free integer order differentiators (Liu, Gibaru, & Perruquetti, 2011; Mboup, Join, & Fliess, 2009) and model-based ones (Fliess & Sira-Ramírez, 2004; Tian, Floquet, & Perruquetti, 2008) for non-linear and linear integer order systems, respectively. More concretely, the latter ones estimate the integer order derivatives of the output of the following integer order linear system:

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = u(t), \tag{1}$$

where y and u are the output and the input, respectively. Recently, these non-asymptotic and robust differentiators have been extended to fractional order case. In Liu et al. (2015), a model-free differentiator has been designed without considering the system model. However, it produces a time-delay in the estimation. In Liu et al. (2015), a model-based differentiator has been proposed to estimate the non-integer order derivatives of the output of the system defined in (1). For the same purpose, another model-based differentiator has been designed by applying the modulating function method in Liu and Laleg-Kirati (2015). The latter two differentiators do not produce any time-delay in estimations. However, they are not applicable to fractional order systems. Having these ideas in mind, the objective of this paper is to extend the algebraic method to design a model-based differentiator to estimate the fractional derivatives of the output of the following fractional order linear system:

$$\sum_{i=0}^{N} a_i \, \mathrm{D}_t^{\alpha_i} y(t) = \sum_{j=0}^{L} b_j \, \mathrm{D}_t^{\gamma_j} u(t), \tag{2}$$

where $\mathrm{D}_t^{\alpha_i} y$ and $\mathrm{D}_t^{\gamma_j} u$ are the fractional derivatives of the output and the input, respectively.

The contributions of this paper can be outlined as follows:

- a digital model-based fractional order differentiator is introduced, which has the following advantages: (i) it can be used to estimate the fractional derivative of the output of an arbitrary order; (ii) it is given by a new algebraic formula, which does not contain any source of errors in continuous noise free case; (iii) it can provide robust estimations in finite-time in discrete noisy cases, without any truncated term error;
- an error bound is provided for the selection of the design parameter;
- there is no need on the initial conditions of the considered system.

This paper is organized as follows: definitions and some useful properties of Riemann–Liouville fractional integrals and derivatives are recalled in Section 2. The main results are given in Section 3. Firstly, the algebraic method is applied to express the fractional derivatives of the output of the considered system by a new algebraic formula in continuous noise free case. Secondly, a digital fractional order differentiator is introduced in discrete noisy cases. Moreover, some error analysis is given. In Section 4, numerical results illustrate the accuracy and the robustness of the proposed fractional order differentiator. Finally, conclusions are summarized in Section 5.

2. Preliminaries

In this section, definitions and some useful properties of fractional integrals and derivatives are recalled. Moreover, some useful formulae related to the Laplace transform are given.

2.1. Riemann–Liouville fractional integrals and derivatives

Let $I = [0, h] \subset \mathbb{R}_+$[2] $\alpha \in \mathbb{R}_+$, and $l = \lceil \alpha \rceil$, where $\lceil \alpha \rceil$ denotes the smallest integer larger than or equal to α. Then, the following definitions are given.

Definition 1 (*Podlubny, 1998, p. 65*)**.** The α order Riemann–Liouville fractional integral of f is defined on $]0, h]$ as follows:

$$\mathrm{D}_t^{-\alpha} f(t) := \begin{cases} f(t), & \text{if } \alpha = 0, \\ \int_0^t \kappa_\alpha(t,\tau) f(\tau)\, d\tau, & \text{else}, \end{cases} \tag{3}$$

where $\kappa_\alpha(t, \cdot)$ is defined by:

$$\kappa_\alpha(t,\tau) := \frac{1}{\Gamma(\alpha)} (t-\tau)^{\alpha-1}, \tag{4}$$

and $\Gamma(\cdot)$ is the well-known Gamma function.

Definition 2 (*Podlubny, 1998, p. 68*)**.** The α order Riemann–Liouville fractional derivative of f is defined on $]0, h]$ as follows:

$$\mathrm{D}_t^{\alpha} f(t) := \frac{d^l}{dt^l} \left\{ \mathrm{D}_t^{\alpha-l} f(t) \right\}. \tag{5}$$

Remark 1. According to (3), if $0 < \alpha < 1$, the Riemann–Liouville fractional integrals are defined by improper integrals. Thus, if $l \neq \alpha$, the Riemann–Liouville fractional derivatives are also defined by improper integrals in (5).

The following formula establishes an additive index law for Riemann–Liouville fractional integrals and derivatives (Podlubny, 1998, pp. 71–74): $\forall n \in \mathbb{N}, \alpha \in \mathbb{R}$,

$$\frac{d^n}{dt^n} \left\{ \mathrm{D}_t^{\alpha} f(t) \right\} = \mathrm{D}_t^{n+\alpha} f(t). \tag{6}$$

In the following lemma, the Leibniz formula for Riemann–Liouville integrals involving a polynomial is recalled. The general Leibniz formula for Riemann–Liouville integrals and derivatives can be found in Diethelm (2010, p. 33) and Miller and Ross (1993, p. 75), respectively.

Lemma 1 (*Miller & Ross, 1993, p. 53*)**.** *Let $\alpha \in \mathbb{R}_+^*$ and $m \in \mathbb{N}$. Then, the following formula holds:*

$$\mathrm{D}_t^{-\alpha} \left[t^m f(t) \right] = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} t^{m-k} \mathrm{D}_t^{-\alpha-k} f(t). \tag{7}$$

2.2. Laplace transform formulae

Let us assume that the Laplace transform of f exists, which is denoted by $\hat{f}$. Then, the Laplace transforms of the Riemann–Liouville

[2] In this paper, $\mathbb{R}_+$ denotes the set of positive real numbers, $\mathbb{R}_+^*$ (resp. $\mathbb{N}^*$) denotes the set of strictly positive real numbers (resp. strictly positive integers), and $\mathbb{Z}_-^*$ denotes the set of strictly negative integers.