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Brief paper

An efficient algorithm for periodic Riccati equation with periodically time-varying input matrix☆

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ABSTRACT

This paper proposes an efficient algorithm for the periodic discrete-time Riccati equation arising from a linear periodic time-varying system $(\mathbf{A}_k, \mathbf{B}_k)$. Although the algorithm can be used for general problem, it is particularly efficient when $\mathbf{A}_k$ is time-invariant and $\mathbf{B}_k$ is periodic, a special property associated with the problem of spacecraft attitude control using magnetic torques. We will compare the proposed algorithm, by using analysis and simulation, to some well-established algorithm to confirm our claim.

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1. Introduction

Riccati equation plays an important role in the linear quadratic regulator (LQR) problem (Lewis, Vrabie, & Syrmos, 2012). For continuous-time linear systems, the optimal solution of the LQR problem is associated with the differential Riccati equation. For discrete-time linear systems, the optimal solution of the LQR is associated with the algebraic Riccati equation. The numerical algorithms for these Riccati equations have been thoroughly studied since the work of Kleinman (1968) and Vaughan (1970). If the linear system is periodic, the optimal solution of the LQR is then associated with the periodic Riccati equation (Bittanti, 1991). For continuous-time periodic linear system, algorithms and solutions of the differential periodic Riccati equation have been studied, for example, in Bittanti, Colaneri, and Guardabassi (1986); Bittanti, Colaneri, and Nicolao (1989) and Varga (2008). For discrete-time periodic linear system, an efficient algorithm was proposed for the algebraic periodic Riccati equation in Hench and Laub (1994).

In this paper, we consider some computer controlled periodic linear system, more specifically, we consider some linear periodic control system $(\mathbf{A}_k, \mathbf{B}_k)$ with $\mathbf{A}_k$ a constant matrix and $\mathbf{B}_k$ a periodic matrix. Hence, the LQR design using the solution of the algebraic periodic Riccati equation is an appropriate selection. Clearly, the algorithm in Hench and Laub (1994) is directly applicable, but it does not make a full use of the property that $\mathbf{A}_k$ is a constant matrix. Therefore, we propose a different algorithm that utilizes this property.

We then consider some real world engineering design problem, spacecraft attitude control using magnetic torques, and show that this problem is indeed a linear periodic control problem with $\mathbf{A}_k$ a constant matrix and $\mathbf{B}_k$ a periodic matrix. Therefore, the proposed algorithm should be a better selection for this problem. We conduct some numerical test and compare the performance of the proposed algorithm with the one in Hench and Laub (1994) to support our claim.

The remainder of the paper is organized as follows. Section 2 describes the LQR design for the linear periodic system with $\mathbf{A}_k$ a constant matrix and $\mathbf{B}_k$ a periodic matrix, and then derives the algorithm for the algebraic periodic Riccati equation. Section 3 shows that the spacecraft attitude control system using magnetic torque is actually a problem as described in Section 2. Section 4 performs some numerical test using both the proposed algorithm and algorithm developed in Hench and Laub (1994) to demonstrate the effectiveness and efficiency of the proposed algorithm. The conclusions are summarized in Section 5.

2. Computational algorithm for the periodic LQR design

In this section, we consider the following discrete-time linear periodic system given as follows:

$$\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{m}_k, \tag{1}$$

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where $\mathbf{A}_k = \mathbf{A}$ is a constant matrix and $\mathbf{B}_k = \mathbf{B}_{k+p}$ is a time-varying periodic matrix. For the discrete-time linear periodic system (1), the LQR state feedback control is to find the optimal $\mathbf{m}_k$ to minimize the following quadratic cost function

$$\lim_{N\to\infty}\left(\min \frac{1}{2}\mathbf{x}_N^{\mathrm{T}}\mathbf{Q}_N\mathbf{x}_N + \frac{1}{2}\sum_{k=0}^{N-1}\mathbf{x}_k^{\mathrm{T}}\mathbf{Q}_k\mathbf{x}_k + \mathbf{m}_k^{\mathrm{T}}\mathbf{R}_k\mathbf{m}_k\right) \tag{2}$$

where

$$\mathbf{Q}_k \geq 0, \tag{3}$$

$$\mathbf{R}_k > 0, \tag{4}$$

and the initial condition $\mathbf{x}_0$ is given. It is well-known that the LQR design for problem (1)–(2) relies on the periodic solution of the periodic Riccati equation (Bittanti, 1991). Our discussion about the computational algorithm is focused on the solution of the algebraic periodic Riccati equation using the special properties of (1), i.e., $\mathbf{A}_k$ is a constant and invertible matrix for all k.

2.1. Preliminary results

Let

$$\mathbf{L} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \in \mathbf{R}^{2n\times 2n}, \tag{5}$$

where n is the dimension of $\mathbf{A}$ or $\mathbf{A}_k$. Note that $\mathbf{L}^{\mathrm{T}} = \mathbf{L}^{-1} = -\mathbf{L}$. Two types of matrices defined below are important to the solutions of the Riccati equations.

Definition 2.1 (*Laub, 1979*). A matrix $\mathbf{M} \in \mathbf{R}^{2n\times 2n}$ is Hamiltonian if $\mathbf{L}^{-1}\mathbf{M}^{\mathrm{T}}\mathbf{L} = -\mathbf{M}$. A matrix $\mathbf{M} \in \mathbf{R}^{2n\times 2n}$ is symplectic if $\mathbf{L}^{-1}\mathbf{M}^{\mathrm{T}}\mathbf{L} = \mathbf{M}^{-1}$.

Proposition 2.1. *If $\mathbf{M}_1$ and $\mathbf{M}_2$ are symplectic, then $\mathbf{M}_1\mathbf{M}_2$ is symplectic.*

Proof. Since $\mathbf{L}^{-1}\mathbf{M}_1^{\mathrm{T}}\mathbf{L} = \mathbf{M}_1^{-1}$ and $\mathbf{L}^{-1}\mathbf{M}_2^{\mathrm{T}}\mathbf{L} = \mathbf{M}_2^{-1}$, we have

$$\begin{aligned}\mathbf{L}^{-1}(\mathbf{M}_1\mathbf{M}_2)^{\mathrm{T}}\mathbf{L} &= \mathbf{L}^{-1}\mathbf{M}_2^{\mathrm{T}}\mathbf{M}_1^{\mathrm{T}}\mathbf{L} = \mathbf{L}^{-1}\mathbf{M}_2^{\mathrm{T}}\mathbf{L}\mathbf{L}^{-1}\mathbf{M}_1^{\mathrm{T}}\mathbf{L} \\ &= \mathbf{M}_2^{-1}\mathbf{M}_1^{-1} = (\mathbf{M}_1\mathbf{M}_2)^{-1}.\end{aligned}$$

This concludes the proof. ■

In the sequel, we use $\lambda(\mathbf{M})$ or simply λ for an eigenvalue of a matrix $\mathbf{M}$ and $\sigma(\mathbf{M})$ for the set of all eigenvalues of $\mathbf{M}$. The following two theorems play essential roles in our discussions.

Theorem 2.1 (*Laub, 1972; Vaughan, 1970*). *Let $\mathbf{M} \in \mathbf{R}^{2n\times 2n}$ is Hamiltonian. Then, $\lambda \in \sigma(\mathbf{M})$ implies $-\lambda \in \sigma(\mathbf{M})$ with the same multiplicity. Let $\mathbf{M} \in \mathbf{R}^{2n\times 2n}$ is symplectic. Then, $\lambda \in \sigma(\mathbf{M})$ implies $\frac{1}{\lambda} \in \sigma(\mathbf{M})$ with the same multiplicity.*

Theorem 2.2 (*Murnaghan & Wintner, 1931*). *Let $\mathbf{M} \in \mathbf{R}^{n\times n}$. Then, there exists an orthogonal similarity transformation $\mathbf{U}$ such that $\mathbf{U}^{\mathrm{T}}\mathbf{M}\mathbf{U}$ is quasi-upper-triangular. Moreover, $\mathbf{U}$ can be chosen such that the 2×2 and 1×1 diagonal blocks appear in any desired order.*

Theorem 2.2 is the so called real Schur decomposition. Combining the above two theorems, we conclude that

Corollary 2.1. *Let $\mathbf{M} \in \mathbf{R}^{2n\times 2n}$ is Hamiltonian or symplectic. Then, there exists an orthogonal similarity transformation $\mathbf{U}$ such that*

$$\begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{bmatrix}^{\mathrm{T}} \mathbf{M} \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{0} & \mathbf{S}_{22} \end{bmatrix} \tag{6}$$

where $\mathbf{U}_{11}, \mathbf{U}_{12}, \mathbf{U}_{21}, \mathbf{U}_{22}, \mathbf{S}_{11}, \mathbf{S}_{12}, \mathbf{S}_{22} \in \mathbf{R}^{n\times n}$, and $\mathbf{S}_{11}$, $\mathbf{S}_{22}$ are quasi-upper-triangular. Moreover,

1. *if $\mathbf{M}$ is Hamiltonian, then $\sigma(\mathbf{S}_{11}) \leq 0$ (or $\sigma(\mathbf{S}_{22}) \geq 0$) and $\sigma(\mathbf{S}_{22}) \geq 0$ (or $\sigma(\mathbf{S}_{11}) \leq 0$).*
2. *if $\mathbf{M}$ is symplectic, then $\sigma(\mathbf{S}_{11})$ lies inside (or outside) the unit circle and $\sigma(\mathbf{S}_{22})$ lies outside (or inside) the unit circle.*

The Hamiltonian matrix is used in the derivation of the solution for the continuous-time differential Riccati equation, while the symplectic matrix is used in the derivation of the solution for the discrete-time algebraic Riccati equation. In our discussion, therefore, the symplectic matrix plays a fundamental role.

2.2. Solution of the algebraic Riccati equation

If $(\mathbf{A}_k, \mathbf{Q}_k)$ is detectable or $\mathbf{Q}_k > 0$, the optimal feedback $\mathbf{m}_k$ is given by Hench and Laub (1994) and Lewis et al. (2012)

$$\mathbf{m}_k = -(\mathbf{R}_k + \mathbf{B}_k^{\mathrm{T}}\mathbf{P}_{k+1}\mathbf{B}_k)^{-1}\mathbf{B}_k^{\mathrm{T}}\mathbf{P}_{k+1}\mathbf{A}_k\mathbf{x}_k, \tag{7}$$

where $\mathbf{P}_k$ is the unique symmetric positive semi-definite solution of the following algebraic Riccati equation (Hench & Laub, 1994; Laub, 1979; Lewis et al., 2012)

$$\begin{aligned}\mathbf{P}_k &= \mathbf{Q}_k + \mathbf{A}_k^{\mathrm{T}}\mathbf{P}_{k+1}\mathbf{A}_k \\ &\quad - \mathbf{A}_k^{\mathrm{T}}\mathbf{P}_{k+1}\mathbf{B}_k(\mathbf{R}_k + \mathbf{B}_k^{\mathrm{T}}\mathbf{P}_{k+1}\mathbf{B}_k)^{-1}\mathbf{B}_k^{\mathrm{T}}\mathbf{P}_{k+1}\mathbf{A}_k,\end{aligned} \tag{8}$$

with the boundary condition $\mathbf{P}_N = \mathbf{Q}_N$. For this algebraic Riccati equation (not necessarily periodic) given as (8), it can be solved using a symplectic system associated with (1) and (2) as follows Hench and Laub (1994), Laub (1979) and Pappas, Laub, and Sandell (1980).

$$\mathbf{E}_k\mathbf{z}_{k+1} = \mathbf{E}_k\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{y}_{k+1} \end{bmatrix} = \mathbf{F}_k\begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} = \mathbf{F}_k\mathbf{z}_k \tag{9}$$

where $\mathbf{x}_k$ is the state and $\mathbf{y}_k$ is the costate,

$$\mathbf{E}_k = \begin{bmatrix} \mathbf{I} & \mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathrm{T}} \\ \mathbf{0} & \mathbf{A}_k^{\mathrm{T}} \end{bmatrix}, \tag{10}$$

$$\mathbf{F}_k = \begin{bmatrix} \mathbf{A}_k & \mathbf{0} \\ -\mathbf{Q}_k & \mathbf{I} \end{bmatrix}. \tag{11}$$

If $\mathbf{A}_k$ is invertible, then $\mathbf{E}_k$ is invertible, and

$$\mathbf{E}_k^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathrm{T}}\mathbf{A}_k^{-\mathrm{T}} \\ \mathbf{0} & \mathbf{A}_k^{-\mathrm{T}} \end{bmatrix}.$$

A related symplectic matrix can be formed (Pappas et al., 1980) as

$$\begin{aligned}\mathbf{Z} &= \mathbf{E}_k^{-1}\mathbf{F}_k = \begin{bmatrix} \mathbf{I} & -\mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathrm{T}}\mathbf{A}_k^{-\mathrm{T}} \\ \mathbf{0} & \mathbf{A}_k^{-\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{A}_k & \mathbf{0} \\ -\mathbf{Q}_k & \mathbf{I} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}_k + \mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathrm{T}}\mathbf{A}_k^{-\mathrm{T}}\mathbf{Q}_k & -\mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathrm{T}}\mathbf{A}_k^{-\mathrm{T}} \\ -\mathbf{A}_k^{-\mathrm{T}}\mathbf{Q}_k & \mathbf{A}_k^{-\mathrm{T}} \end{bmatrix}.\end{aligned} \tag{12}$$

It is straightforward to verify that $\mathbf{L}^{-1}\mathbf{Z}^{\mathrm{T}}\mathbf{L} = \mathbf{Z}^{-1}$; therefore, from Corollary 2.1, there exists an orthogonal matrix $\mathbf{U}$ such that

$$\begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{bmatrix}^{\mathrm{T}} \mathbf{Z} \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{0} & \mathbf{S}_{22} \end{bmatrix}. \tag{13}$$

The (steady state) solution of (8) is given as follows Laub (1979, Theorem 6).

Theorem 2.3. *$\mathbf{U}_{11}$ is invertible and $\mathbf{P} = \mathbf{U}_{12}\mathbf{U}_{11}^{-1}$ solves (8) with $\mathbf{P} = \mathbf{P}^{\mathrm{T}} \geq 0$;*

$$\begin{aligned}\sigma(\mathbf{S}_{11}) &= \sigma(\mathbf{A}_k - \mathbf{B}_k(\mathbf{R}_k + \mathbf{B}_k^{\mathrm{T}}\mathbf{P}_k\mathbf{B}_k)^{-1}\mathbf{B}_k^{\mathrm{T}}\mathbf{P}_k\mathbf{A}_k) \\ &= \sigma(\mathbf{A}_k - \mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathrm{T}}\mathbf{A}_k^{-\mathrm{T}}(\mathbf{P}_k - \mathbf{Q}_k)) \\ &= \sigma(\mathbf{A}_k - \mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathrm{T}}(\mathbf{P}_k^{-1} - \mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^{\mathrm{T}})^{-1}\mathbf{A}_k) \\ &= \text{the “closed-loop” spectrum}.\end{aligned} \tag{14}$$