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Local observability of nonlinear positive continuous-time systems☆

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ABSTRACT

Two observability properties of continuous-time nonlinear positive systems are studied: local observability and local positive observability. An algebraic necessary and sufficient condition for local observability is proved. It is expressed with the aid of ideals and cones in the ring of germs of analytic functions. Local positive observability is characterized in a more geometric fashion. This characterization naturally extends the known characterization of positive observability of positive linear systems. As for linear systems, criteria for local positive observability for nonlinear systems are very restrictive. This result is parallel to characterization of local positive reachability of nonlinear systems.

1. Introduction

One of the important properties of linear systems that have different characterizations for discrete-time and continuous-time cases is positive observability. A positive discrete-time linear system can be positively observable even if there is only one output variable (see e.g. Farina & Rinaldi, 2000; Kaczorek, 2002). On the other hand, to be positively observable, the positive continuous-time system $\dot{x} = Ax$, $y = Cx$, must have at least n output variables, where n is the dimension of the state space. Moreover, the matrix C has to contain an $n \times n$ submatrix whose inverse exists and is nonnegative and the matrix A must be diagonal (Bartosiewicz, 2012). This result is dual to a characterization of positive reachability of positive continuous-time systems in Commault and Alamir (2007) and Valcher (2009). Similarly, positive reachability of discrete-time linear systems as dual to positive observability is much less restrictive than in the continuous-time case (Farina & Rinaldi, 2000; Kaczorek, 2002). In Bartosiewicz (2012) positive observability of linear systems on time scales was characterized with the aid of modified Gram matrices. This tool allowed to unify very different criteria for positive observability of discrete- and continuous-time linear systems.

The goal of this paper is to examine observability properties of nonlinear positive systems. We concentrate on two types of local observability: ordinary one but with the restriction to the positive cone of the state space, and positive local observability, which is a local version of positive observability studied for linear systems in Bartosiewicz (2012). Local observability is a fundamental system property for nonlinear systems (Hermann & Krener, 1977). But due to the restriction to the positive cone it has slightly different properties from the unrestricted case, especially when the point of the state space at which local observability is considered lies on the boundary of the cone. Local positive observability is a stronger property than local observability and could be more useful in practice. However, as we indicate later, because of its very restrictive nature, its use is limited. For simplicity we restrict the class of systems to systems without control of the form $\dot{x} = f(x)$, $y = h(x)$. To make the most of the results of real analytic geometry we assume that f and h are analytic. This will be especially useful for establishing necessary and sufficient conditions for local observability. The main difference between local observability of systems without positivity assumption and local observability of positive systems lies in the type of neighborhood of the point at which the observability is studied. For positive systems, the point belongs to the positive cone and by its neighborhood we mean the intersection of an open neighborhood and the positive cone. This results in modification of the necessary and sufficient condition for local observability given in Bartosiewicz (1995). To address the inequalities defining the cone, we exploit one of the fundamental results of real analytic geometry: *Positivstellensatz*, which is a generalization of *Nullstellensatz* used in Bartosiewicz (1995). Local observability of nonlinear systems without positivity assumption was studied by many authors (see e.g. Isidori, 1995; Nijmeijer & van der Schaft, 1990 and the references therein). In practical applications, the sufficient condition for local observability, so

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called rank condition developed in Hermann and Krener (1977), has usually been applied.

The main contribution of this paper is a characterization of local positive observability of positive nonlinear systems. This property means that one can recover the state of a positive system from the output in a positive way (through a nonlinear positive map). The precise definition is given in Section 2. We show that such a system is locally positively observable at 0 if and only if the vector field f is tangent to all the subfaces of the positive cone and h is a local diffeomorphism preserving skeletons of the positive cone of all dimensions. Specifying these conditions for a linear system we can recover the criteria for positive observability found in Bartosiewicz (2012). Criteria for local positive observability for nonlinear systems are as restrictive as in the linear case. In particular, it is necessary that the number of output variables is greater than or equal to the number of state variables. Thus this result is in some sense negative: the structure of a positive system inhibits effective positive observability. This is parallel to local positive reachability of nonlinear positive systems. It holds only if the number of controls is greater than or equal to the number of states (Bartosiewicz, 2016). On the other hand, positive reachability or observability of discrete-time systems is much less restrictive. It can hold for systems with one input or output variable. So our result helps understand the differences between continuous- and discrete-time systems.

2. Preliminaries

Let us first set notation and terminology. By $\mathbb{R}_+$ we denote the set of all nonnegative real numbers. Similarly, $\mathbb{R}^n_+$ and $\mathbb{R}^{n\times m}_+$ denote real n-dimensional column vectors and $n\times m$ matrices with nonnegative elements. A column or row vector is called *i-monomial* if its ith component is positive and the others are 0. It is called *monomial*, if it is i-monomial for some i. The i-monomial column vector whose ith component is equal 1 is denoted by e_i. An $n \times n$ real matrix is *monomial*, if all its rows and columns are monomial. Then such a matrix is invertible and its inverse is also monomial. If X is an arbitrary set, then a map $f : X \to \mathbb{R}^n_+$ is *i-monomial*, if for every $x \in X$ the vector $f(x)$ is i-monomial.

By a cone in $\mathbb{R}^n$ we mean a subset K of $\mathbb{R}^n$ such that if $x \in K$ then for every $\alpha \in \mathbb{R}_+, \alpha x \in K$. The set $\mathbb{R}^n_+$ is a cone. It has n *faces*. Each face is the intersection of $\mathbb{R}^n_+$ with some hyperplane $x_i = 0$, where $i \in \{1,\dots,n\}$. The faces are cones as well. For a nonempty subset I of $\{1,\dots,n\}$ let K_I denote the intersection of $\mathbb{R}^n_+$ with all the hyperplanes $x_i = 0$ with $i \notin I$. Each K_I is a cone. It will be called a subface of $\mathbb{R}^n_+$ of dimension $|I|$, where $|I|$ denotes the cardinality of I. If $I = \{i\}$, then K_I is the nonnegative half-axis $x_i \geq 0$. It will be shortly denoted by K_i. The *k-dimensional skeleton of* $\mathbb{R}^n_+$, denoted by S_k, is the union of all k-dimensional subfaces of $\mathbb{R}^n_+$. Let U be an open neighborhood of 0. We say that a vector field f on $\mathbb{R}^n$ *is tangent to* $K_I \cap U$ if for all $x \in K_I \cap U, f_j(x) = 0$ for $j \notin I$. For $I = \{1,\dots,n\}$, $K_I = \mathbb{R}^n_+$.

By a *nonnegative neighborhood of* 0 in $\mathbb{R}^n$ we mean the intersection of some open neighborhood of 0 with $\mathbb{R}^n_+$. If $A \subset \mathbb{R}^n$, then ∂A denotes the boundary of A. Observe that the boundary of $\mathbb{R}^n_+$ coincides with the skeleton of dimension $n-1$, i.e. $\partial\mathbb{R}^n_+ = S_{n-1}$.

Let U be an open subset of $\mathbb{R}^n$, such that $U \cap \mathbb{R}^n_+$ is nonempty. A map $g : U \to \mathbb{R}^n$ is called *positive* if $f(U \cap \mathbb{R}^n_+) \subset \mathbb{R}^n_+$.

We shall study a nonlinear system of the form

$$\dot{x}(t) = f(x(t)), \qquad y(t) = h(x(t)), \tag{1}$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$. We assume that f is an analytic vector field on $\mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ is an analytic map. Let $\gamma(t, x_0)$ denote the trajectory of f corresponding to the initial condition $x(0) = x_0$ and evaluated at time $t \geq 0$. Note that γ may be not defined for all $t \geq 0$.

The points x_1 and x_2 from $\mathbb{R}^n$ are called *indistinguishable* (by system (1)) if $h(\gamma(t,x_1)) = h(\gamma(t,x_2))$ for all $t \geq 0$ for which both $\gamma(t,x_1)$ and $\gamma(t,x_2)$ are defined.

Recall that the Lie derivative of a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ with respect to a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is the function $L_f\varphi = \varphi' \cdot f$, where $\varphi' = (\frac{\partial\varphi}{\partial x_1},\dots,\frac{\partial\varphi}{\partial x_n})$ is the gradient of φ. Let $\mathcal{H}$ be a family of functions on $\mathbb{R}^n$ consisting of all the components h_i of the map h and all their Lie derivatives $L^k_f h_i$, where $k \in \mathbb{N}$ and $i = 1,\dots,p$. We shall often identify h_i with $L^0_f h_i$.

Theorem 1 (*Hermann & Krener, 1977*)**.** *The points x_1 and x_2 are indistinguishable if and only if for all $\varphi \in \mathcal{H}$, $\varphi(x_1) = \varphi(x_2)$.*

We say that system (1) is *positive* if for every $x_0 \in \mathbb{R}^n_+$, $x(t) = \gamma(t,x_0) \in \mathbb{R}^n_+$ and $y(t) = h(\gamma(t,x_0)) \in \mathbb{R}^p_+$ for every $t \geq 0$ for which $\gamma(t,x_0)$ is defined.

Theorem 2 (*See e.g. Bartosiewicz, 2015*)**.** *System (1) is positive if and only if for every $i \in \{1,\dots,n\}$ and for every x belonging to the face $x_i = 0$, $f_i(x) \geq 0$, where f_i is the ith component of f, and for every $x \in \mathbb{R}^n_+$, $h(x) \in \mathbb{R}^p_+$.*

We define now several concepts of observability for positive systems.

System (1) is *observable on* $\mathbb{R}^n_+$ if any two distinct points from $\mathbb{R}^n_+$ are distinguishable.

Let $x_0 \in \mathbb{R}^n_+$. System (1) is *locally observable at* x_0 if there is an open neighborhood U of x_0 such that for any $\bar{x} \in \mathbb{R}^n_+ \cap U$ the points x_0 and $\bar{x}$ are distinguishable.

System (1) is *positively observable* if there is $T > 0$ and an analytic map $\Phi : [0,T] \times \mathbb{R}^p_+ \to \mathbb{R}^n_+$ such that for any $\bar{x} \in \mathbb{R}^n_+$, $\bar{x} = \int_0^T \Phi(t, h(\gamma(t,\bar{x})))dt$.

System (1) is *locally positively observable at* x_0 if there is an open neighborhood U of x_0, $T > 0$ and an analytic map $\Phi : [0,T] \times \mathbb{R}^p_+ \to \mathbb{R}^n_+$ such that for any $\bar{x} \in \mathbb{R}^n_+ \cap U$, $\bar{x} = \int_0^T \Phi(t, h(\gamma(t,\bar{x})))dt$.

Clearly, observability on $\mathbb{R}^n_+$ implies local observability at any $x_0 \in \mathbb{R}^n_+$ and positive observability implies local positive observability at any $x_0 \in \mathbb{R}^n_+$. Moreover local positive observability at x_0 implies local observability at x_0. Though all these properties are meaningful for any system of form (1), we shall concentrate on positive systems.

If x_0 belongs to the interior of $\mathbb{R}^n_+$, local observability at x_0 does not differ from the standard concept, where the system is considered on an open subset of $\mathbb{R}^n$ or an analytic manifold. Let us recall an algebraic characterization of this property.

The definitions of the following algebraic and geometric notions and *Positivstellensatz* can be found e.g. in Andradas, Bröcker, and Ruiz (1996). If I is an ideal of a commutative ring R, then the *real radical of* I, denoted by $\sqrt[\mathbb{R}]{I}$, is the subset of R consisting of elements a for which there are $k \geq 0$, $m \geq 1$ and $b_1,\dots,b_k \in R$ such that $a^{2m} + b_1^2 + \dots + b_k^2 \in I$. The ideal of R generated by $c_1,\dots,c_r \in R$ is denoted by $(c_1,\dots,c_r)$.

Let $\mathcal{A}_{x_0}$ denote the ring of germs of analytic functions at x_0. Such germs may be identified with convergent power series centered at x_0. Let I_{x_0} be the ideal of $\mathcal{A}_{x_0}$ generated by the germs at x_0 of functions of the form $\varphi - \varphi(x_0)$, where $\varphi \in \mathcal{H}$. Let m_{x_0} be the maximal ideal of $\mathcal{A}_{x_0}$. It is generated by germs of functions $x_i - x_{0i}$, $i = 1,\dots,n$.

For x_0 belonging to the interior of $\mathbb{R}^n_+$ we have:

Theorem 3 (*Bartosiewicz, 1995*)**.** *System (1) is locally observable at x_0 if and only if $\sqrt[\mathbb{R}]{I_{x_0}} = m_{x_0}$.*

However, if x_0 belongs to the boundary of $\mathbb{R}^n_+$, then the theorem is no longer true. We solve this problem in the next section.

Let $x_0 \in \mathbb{R}^n$. A subset K of the ring $\mathcal{A}_{x_0}$ is called a *cone* if it satisfies the following conditions: