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Auxiliary function-based summation inequalities and their applications to discrete-time systems ☆

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ABSTRACT

Auxiliary function-based summation inequalities are addressed in this technical note. By constructing appropriate auxiliary functions, several new summation inequalities are obtained. A novel sufficient criterion for asymptotic stability of discrete-time systems with time-varying delay is obtained in terms of linear matrix inequalities. The advantage of the proposed method is demonstrated by two classical examples from the literature.

1. Introduction

The stability of discrete-time systems with time-varying delay has attracted much attention. Many stability criteria have been derived for time-delay systems via different techniques (see Feng, 2002; Feng, Lam, & Yang, 2015; Kwon, Park, Ju, Lee, & Cha, 2013; Liu & Zhang, 2012; Ramakrishnan & Ray, 2013; Zhang & Han, 2015a; Zhang, Xu, & Zou, 2008). In order to estimate the bounds of terms such as $\int_a^b x^T(s)Px(s)ds$ (or $\sum_{i=a}^{b} x^T(i)Px(i)$) which often occur in the derivative (or the difference) of Lyapunov functionals, Jensen's inequality has been extensively used, since it is an appropriate tool to derive tractable stability conditions expressed in terms of linear matrix inequalities (LMIs).

In Seuret and Gouaisbaut (2013), an improved Jensen's inequality called the Wirtinger-based integral inequality was proposed. Park, Lee, and Lee (2015) proved an auxiliary function-based integral inequality which extended the Wirtinger-based integral inequality. Zhang and Han (2015b) established an Abel lemma-based finite-sum inequality which can be regarded as a discrete counterpart of the Wirtinger-based integral inequality, and the stability of linear discrete systems with constant delay was investigated.

Motivated by the ideas above, this paper establishes some new auxiliary function-based summation inequalities which improve upon the Abel lemma-based finite-sum inequality. Unlike the ideas in Park et al. (2015) and Zhang and Han (2015b), an orthogonal group of sequences is chosen from an orthogonal system of polynomials, which can be regarded as an orthogonal system of continuous functions. Another orthogonal group of sequences is chosen from another orthogonal system, which can be regarded as an orthogonal system of discontinuous functions. Based on the idea of approximation of a vector in these two orthogonal systems, a new summation inequality is obtained. This new summation inequality improves and extends the Abel lemma-based finite-sum inequality and the discrete counterpart of the integral inequality in Park et al. (2015). Moreover, as an application of these inequalities, a new less conservative asymptotic stability criterion is provided here for discrete-time systems with time-varying delay. Two numerical examples are provided to illustrate the effectiveness of the proposed method.

2. Main results

Theorem 1. *Given an $n \times n$ positive definite matrix $R > 0$, any sequence of a discrete-time variable $y : [-h, 0] \cap Z \to R^n$ and three sequences $p_i : [-h, 0] \cap Z \to R$, $i = 1, 2, 3$ satisfying $\sum_{k=-h+1}^{0} p_i(k) = 0$, $\sum_{k=-h+1}^{0} p_1(k)p_2(k) = 0$ and $\sum_{k=-h+1}^{0} p_2(k)p_3(k) = 0$, the following inequality holds:*

$$\sum_{i=-h+1}^{0} y(i)^T R y(i) \geq \frac{1}{h}\Theta_0^T R \Theta_0 + \sum_{i=1}^{3} \frac{1}{P_i}\Theta_i^T R \Theta_i, \tag{1}$$

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where $\Theta_0 = \sum_{k=-h+1}^{0} y(k)$, $\Theta_1 = F_1, \Theta_2 = F_2, \Theta_3 = F_3 - \frac{F_1}{P_1}\sum_{i=-h+1}^{0} p_1(i)p_3(i)$, $F_j = \sum_{i=-h+1}^{0} y(i)p_j(i)$, $P_j = \sum_{i=-h+1}^{0} p_j^2(i)$, $j = 1, 2, 3$.

Proof. Consider the following energy function

$$J(v) = \sum_{i=-h+1}^{0} z(i,v)^T R z(i,v), \tag{2}$$

where $z(i,v) = y(i) - \frac{1}{h}\Theta_0 - \frac{p_1(i)}{P_1}F_1 - \frac{p_2(i)}{P_2}F_2 - p_3(i)v$ and $v \in R^n$.

Let $\nabla J(\hat{v}) = 0$, where ∇ is the gradient operator. Then $\hat{v} = \frac{F_3}{P_3} - \frac{F_1}{P_1P_3}\sum_{i=-h+1}^{0} p_1(i)p_3(i)$. Since $\nabla^2 J(\hat{v}) = 2\sum_{i=-h+1}^{0} p_3^2(i)R > 0$, then the Hessian matrix of $J(v)$ at $v = \hat{v}$ is positive definite. Thus, the energy function $J(v)$ reaches its minimum when $v = \hat{v}$. Furthermore,

$$J(\hat{v}) = \sum_{i=-h+1}^{0} y(i)^T R y(i) - \frac{1}{h}\Theta_0^T R \Theta_0 - \frac{1}{P_1}F_1^T R F_1 - \frac{1}{P_2}F_2^T R F_2 - P_3\hat{v}^T R \hat{v}. \tag{3}$$

The nonnegativity of $J(\hat{v})$ implies

$$\sum_{i=-h+1}^{0} y(i)^T R y(i) \geq \frac{1}{h}\Theta_0^T R \Theta_0 + \frac{1}{P_1}F_1^T R F_1 + \frac{1}{P_2}F_2^T R F_2 + P_3\hat{v}^T R \hat{v}. \tag{4}$$

Substituting $\hat{v}$ into Inequality (4) yields Inequality (1). This completes the proof of Theorem 1.

If we choose the following sequences for $p_j(i)$ in Theorem 1

$$p_1(i) = h - 1 + 2i, \qquad p_2(i) = i^2 + (h-1)i + \frac{(h-1)(h-2)}{6},$$

$$p_3(i) = \begin{cases} -1, & i = 0, \\ 0, & -h+2 \leq i \leq -1, \\ 1, & i = -h+1, \end{cases}$$

then

$$\sum_{i=-h+1}^{0} p_m(i) = 0, \quad m = 1, 2, 3,$$

$$\sum_{i=-h+1}^{0} p_1(i)p_2(i) = \sum_{i=-h+1}^{0} p_2(i)p_3(i) = 0,$$

$$\sum_{i=-h+1}^{0} p_1(i)p_3(i) = -2(h-1),$$

$$P_1 = \sum_{i=-h+1}^{0} p_1^2(i) = \frac{(h-1)h(h+1)}{3},$$

$$P_2 = \sum_{i=-h+1}^{0} p_2^2(i) = \frac{(h-2)(h-1)h(h+1)(h+2)}{180},$$

$$P_3 = \sum_{i=-h+1}^{0} p_3^2(i) = 2.$$

After some simple computations, one gets

$$\sum_{i=-h+1}^{0} p_1(i)y(i) = -(h+1)\left[\sum_{s=-h+1}^{0} y(s) - \frac{2}{h+1}\sum_{k=-h+1}^{0}\sum_{s=k}^{0} y(s)\right], \tag{5}$$

$$\sum_{i=-h+1}^{0} p_2(i)y(i) = \frac{(h+1)(h+2)}{6}\left[\sum_{i=-h+1}^{0} y(i) - \frac{6}{h+1}\sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k) + \frac{12}{(h+1)(h+2)}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\sum_{m=k}^{0} y(m)\right], \tag{6}$$

and

$$\sum_{i=-h+1}^{0} p_3(i)y(i) = -y(0) + y(-h+1). \tag{7}$$

By Theorem 1, one now gets the following corollary:

Corollary 1. *Given a positive definite matrix $R > 0$, any sequence of a discrete-time variable $y : [-h, 0] \cap Z \to R^n$, and any positive integer $h > 2$, the following inequality holds:*

$$\sum_{i=-h+1}^{0} y(i)^T R y(i) \geq \frac{1}{h}\Theta_0^T R \Theta_0 + \frac{3(h+1)}{h(h-1)}\Omega_1^T R \Omega_1 + \frac{5(h+1)(h+2)}{(h-2)(h-1)h}\Omega_2^T R \Omega_2 + \frac{1}{2}\Omega_3^T R \Omega_3, \tag{8}$$

where $\Theta_0 = \sum_{i=-h+1}^{0} y(i)$, $\Omega_1 = \Theta_0 - \frac{2}{h+1}\sum_{i=-h+1}^{0}\sum_{j=i}^{0} y(j)$, $\Omega_2 = \Theta_0 + \frac{12}{(h+1)(h+2)}\sum_{i=-h+1}^{0}\sum_{j=i}^{0}\sum_{k=j}^{0} y(k) - \frac{6}{h+1}\sum_{i=-h+1}^{0}\sum_{j=i}^{0} y(j)$, $\Omega_3 = y(-h+1) - y(0) - \frac{6}{h}\Omega_1$.

Setting $y(i) = \triangle x(i) = x(i) - x(i-1)$ in Corollary 1, we have

Corollary 2. *Given a positive definite matrix $R > 0$, any sequence of a discrete-time variable $x : [-h, 0] \cap Z \to R^n$, and any positive integer $h > 2$, the following inequality holds:*

$$\sum_{i=-h+1}^{0} \triangle x(i)^T R \triangle x(i) \geq \frac{1}{h}\Theta_{\triangle 0}^T R \Theta_{\triangle 0} + \frac{3(h+1)}{h(h-1)}\Omega_{\triangle 1}^T R \Omega_{\triangle 1} + \frac{5(h+1)(h+2)}{(h-2)(h-1)h}\Omega_{\triangle 2}^T R \Omega_{\triangle 2} + \frac{1}{2}\Omega_{\triangle 3}^T R \Omega_{\triangle 3}, \tag{9}$$

where $\Theta_{\triangle 0} = x(0) - x(-h)$, $\Omega_{\triangle 1} = x(0) + x(-h) - \frac{2}{h+1}\sum_{k=-h}^{0} x(k)$, $\Omega_{\triangle 2} = x(0) - x(-h) + \frac{6h}{(h+1)(h+2)}\sum_{i=-h}^{0} x(i) - \frac{12}{(h+1)(h+2)}\sum_{i=-h+1}^{0}\sum_{k=i}^{0} x(k)$, $\Omega_{\triangle 3} = x(-h+1) + \frac{6-h}{h}x(-h) + \frac{6-h}{h}x(0) + x(-1) - \frac{12}{h(h+1)}\sum_{i=-h}^{0} x(i)$.

Remark 1. In Park et al. (2015), $p_1(s)$ and $p_2(s)$ must be chosen from an orthogonal system of continuous functions. In this paper, a further function $p_3(s)$ is added, which can be chosen from another orthogonal system of discontinuous functions. $p_3(s)$ need not be orthogonal to $p_1(s)$. The complexity of computation incurred by use of orthogonal polynomials with high degree is avoided. The introduction of $p_3(s)$ leads to a sharper inequality. Based on this idea for the discrete case, an orthogonal group of sequences $\{1, p_1(k), p_2(k)\}$ is chosen from an orthogonal system of continuous functions. $p_3(k)$ is chosen from another orthogonal system of functions, where $p_3(s)$ may be discontinuous. By approximating vector $y(i)$, a new summation inequality is obtained in Theorem 1. If we choose appropriate sequences $p_i(k)$, $i = 1, 2, 3$ as in Corollary 2, then the $\Omega_{\triangle i}$, $i = 1, 2, 3$ are very simple, and suitable for stability analysis of discrete-time systems with