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Asymptotic stabilization of submanifolds embedded in Riemannian manifolds☆

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ABSTRACT

We study control problems in which systems whose states spaces are Riemannian manifolds must be steered towards embedded submanifolds of their state spaces in a stable fashion, i.e., feedback laws rendering the given submanifold asymptotically stable are sought. Bringing the closed loop to the form of a gradient system with drift, we find that the gradient part can be scaled in a fashion so as to guarantee asymptotic stability of the desired submanifold. Under this circumstance, we show that solutions of the system can be brought to an arbitrarily small neighborhood of the submanifold in arbitrary time. Thereafter, we derive an algebraic condition under which the control vector fields can be brought to the desired form. Lastly, we recast our algebraic condition in terms of vertical and horizontal spaces for the case that the submanifold is an equivalence class.

1. Introduction

Recent years have witnessed an increasing interest in control problems which cannot be formulated as setpoint regulation problems, such as synchronization, path following, or pattern generation, to name a few. A commonality of these problems is that they admit formulation as submanifold stabilization problems, where the submanifold is the diagonal of a product space for synchronization (cf. Hale, 1997), the image of a curve for path following (cf. Nielsen & Maggiore, 2006), and some homotopy circle (i.e. a submanifold homotopy equivalent to the circle) for pattern generation (cf. Brockett, 2003). In the light of this observation, broadening our focus and constructing solutions to general submanifold stabilization problems admits for application to all of the aforementioned problems, where the classical setpoint regulation problem would be the trivial case of the submanifold being a singleton.

For submanifolds of $\mathbb{R}^n$, the asymptotic submanifold stabilization problem was approached via the backstepping technique (El-Hawwary & Maggiore, 2010), tools from passivity (El-Hawwary & Maggiore, 2013; Montenbruck, Arcak, & Allgöwer, 2015), (transverse) feedback linearization (Nielsen & Maggiore, 2008), and gradient vector fields (Montenbruck, Bürger, & Allgöwer, 2016). Topological obstructions to submanifold stabilization were studied in Mansouri (2007) and Mansouri (2010).

For submanifolds of state spaces which are no vector spaces, the asymptotic submanifold stabilization problem has, to our knowledge, so far only been studied for the synchronization problem: Synchronization was studied on the circle (Kuramoto, 1975; Scardovi, Sarlette, & Sepulchre, 2007), on the orthogonal group (Sarlette, Sepulchre, & Leonard, 2009), on general Lie groups (Sarlette, Bonnabel, & Sepulchre, 2010), and on rather general Riemannian manifolds (Montenbruck, Bürger, & Allgöwer, 2015; Sarlette & Sepulchre, 2009; Tron, Afsari, & Vidal, 2013).

General asymptotic submanifold stabilization problems on Riemannian manifolds have not yet been studied, but are in the focus of the present paper. This is motivated, e.g., by oscillators, which live on homotopy circles, rigid bodies, which live on the Euclidean group, or rotations, which live on the orthogonal group. As in Montenbruck et al. (2016), we employ gradient vector fields in our analysis. Apart from the extension of these ideas to Riemannian manifolds, we herein present an algebraic characterization of asymptotic stabilizability which we further recast in terms of horizontal and vertical spaces for the case that the submanifold belongs to some quotient manifold.

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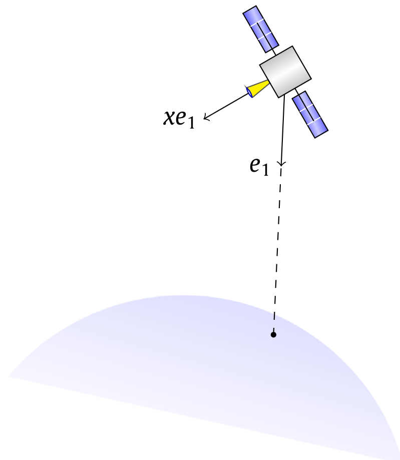

Fig. 1. Letting the rotation of a satellite be $x \in \mathrm{SO}(3)$ and letting its telescope be mounted along the body-fixed axis e_1, facing a point on earth, say e_1, is the task of aligning xe_1 with e_1 by appropriate choice of x.

2. Motivating example

Consider a reconnaissance satellite with which we must perform a surveillance task. Modeling the satellite as a rigid body, its rotation x is a member of the special orthogonal group $\mathrm{SO}(3)$, a Riemannian manifold. In a surveillance task, the satellite must face a point (say face a point on earth with a telescope). However, there are multiple rotations for which the satellite faces this point. More particular, but without losing generality, let the point to be faced by the satellite be the unit vector

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

and let the telescope be mounted along the body-fixed coordinate e_1. Then all solutions in x to the linear equation

$$xe_1 = e_1 \tag{2}$$

solve the surveillance task; this setup is sketched in Fig. 1.

Parameterizing x with Euler angles, i.e.

$$x = \mathrm{rot}_1(\alpha_1)\,\mathrm{rot}_3(\alpha_2)\,\mathrm{rot}_1(\alpha_3), \tag{3}$$

$$\mathrm{rot}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \tag{4}$$

$$\mathrm{rot}_3(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

one finds that (2) is equivalent to $\cos(\alpha_2) = 1$, i.e. $\mathrm{rot}_3(\alpha_2) = I$, where I denotes the identity of $\mathrm{SO}(3)$. This lets solutions to (2) in x simplify to

$$x = \mathrm{rot}_1(\alpha_1)\,\mathrm{rot}_1(\alpha_3) = \mathrm{rot}_1(\alpha_1 + \alpha_3), \tag{6}$$

i.e. the one-dimensional submanifold

$$S = \mathrm{rot}_1([0, 2\pi]) \subset \mathrm{SO}(3) \tag{7}$$

of $\mathrm{SO}(3)$ contains the solutions to the surveillance task.

If one now had to find a controller that solves the surveillance task, one could, of course, asymptotically stabilize any point on S to solve the problem. The situation however becomes more delicate when it is not possible to asymptotically stabilize singletons on S, for instance due to a drift vector field, say f, nonvanishing on S, or a missing control input tangent to S (for the latter, cf. Byrnes & Isidori, 1991). Such drift could, e.g., be caused by solar (radiation) pressure. For instance, let the kinematic equations describing the motion of the satellite be given by

$$\dot{x} = f(x) + g_1(x)\,u_1(x) + g_2(x)\,u_2(x) \tag{8}$$

where $u_1, u_2 : \mathrm{SO}(3) \to \mathbb{R}$ are the controls sought, f is the drift vector field

$$f(x) = x\Omega_1 \tag{9}$$

and g_1, g_2 are the control vector fields

$$g_1(x) = x\Omega_2 \tag{10}$$

$$g_2(x) = x\Omega_3 \tag{11}$$

wherein $\Omega_1, \Omega_2, \Omega_3$ are the infinitesimal generators

$$\Omega_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \Omega_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

of the Lie algebra of $\mathrm{SO}(3)$, such that S is reachable (cf. Brockett, 1972). However, no singleton contained in S can be stabilized asymptotically in order to solve the surveillance task (this is due to the fact that one has no influence in the direction of f but the drift acts in this very direction). We thus decide to asymptotically stabilize S itself: this solves the surveillance task as well as asymptotically stabilizing a singleton on S, which we have seen by characterizing the solutions to (2). One possible approach to solve this problem is to find a scalar field φ on $\mathrm{SO}(3)$ which vanishes critically on S but which is positive and regular on $U \setminus S$, where U is an appropriate neighborhood of S. If this function, in addition, is strongly convex on the images of the normal exponential map (we study these properties more carefully in Section 5), then its gradient vector field $\mathrm{grad}\,\varphi$ can be used to asymptotically stabilize S via solving

$$g_1 u_1 + g_2 u_2 = -k\,\mathrm{grad}\,\varphi \tag{13}$$

for u_1, u_2 (assuming that such a solution exists), for some sufficiently large k. In the present example, we could, for instance, choose

$$\varphi(x) = -\frac{1}{2}\,\mathrm{tr}\left(\log\left(\mathrm{rot}_1(-\alpha_3)\,\mathrm{rot}_3(-\alpha_2)\,\mathrm{rot}_1(\alpha_3)\right)^2\right) \tag{14}$$

(with the convention that x equals $\mathrm{rot}_1(\alpha_1)\,\mathrm{rot}_3(\alpha_2)\ \ \mathrm{rot}_1(\alpha_3)$), where log is the logarithmic map, bringing elements of $\mathrm{SO}(3)$ to elements of its Lie algebra. Simply speaking, φ measures the squared distance of $\mathrm{rot}_3(\alpha_2)$ to I (realized by measuring the squared distance of x to some representative, $\mathrm{rot}_1(\alpha_1 + \alpha_3)$, of x in S), which in turn measures squared the distance of x to S. The gradient vector field of φ is given by

$$\mathrm{grad}\,\varphi(x) = -x\log\left(\mathrm{rot}_1(-\alpha_3)\,\mathrm{rot}_3(-\alpha_2)\,\mathrm{rot}_1(\alpha_3)\right), \tag{15}$$

which is a linear combination of g_1 and g_2 ($\varphi(x)$ remains constant as α_1 changes) and thus admits to solve (13) for u_1, u_2. Applying the resulting controls to (8) reveals that S is asymptotically stabilized for $k > 0$. This is best depicted by solving $\dot{x} = f(x) - \mathrm{grad}\,\varphi(x)$ for some initial condition $x(0) = x_0$ and plotting $x([0,\infty))\,e_1$, $x([0,\infty))\,e_3$, where

$$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{16}$$

which is shown in Fig. 2. We can see that xe_1 approaches e_1, as desired in order to solve the surveillance task, whilst xe_3 approaches a unit circle contained in $\mathrm{span}(\{e_2, e_3\})$, where

$$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{17}$$

which is due to the drift vector field f, i.e. the satellite asymptotically faces the desired point but rotates around the telescope axis.