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New cut-balance conditions in networks of clusters ☆

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ABSTRACT

Existing results in the literature guarantee that the state of multi-agent systems interacting over networks that satisfy the cut-balance assumption asymptotically converges to a constant vector. Furthermore, when the network is persistently connected the agents reach a common value called consensus. Many real large-scale networks are obtained by sparsely connecting subnetworks of densely connected agents. In this context, our objective is to provide new cut-balance assumptions that are adapted to networks of clusters. They are useful for consensus and agreement in clusters in situations when network topology is such that clusters are given or can be easily identified. In this case our new cut-balance assumptions can be checked by realizing a smaller number of operations.

1. Introduction

The multi-agent framework is widely used to model the dynamics of large numbers of interconnected systems. The most studied problem in this context is the consensus or synchronization of all agents in the network. The convergence to consensus is typically characterized by conditions that depend on the communication graph between agents. Basic results concern fixed undirected topologies but notable advances towards directed and time varying topologies have been provided in Hendrickx and Tsitsiklis (2013), Jadbabaie, Lin, and Morse (2003), Moreau (2005) and Ren and Beard (2005) for discrete time dynamics and (Hendrickx & Tsitsiklis, 2013; Martin & Girard, 2013; Martin & Hendrickx, 2016; Olfati-Saber & Murray, 2004; Ren & Beard, 2005) for continuous time algorithms.

In Hendrickx and Tsitsiklis (2013), the authors introduced the assumption of cut-balance communication which is a general form of communication reciprocity among the agents. Under the cut-balance assumption, convergence is ensured, and consensus may occur in groups or globally. The cut-balance assumption was extended in Martin and Girard (2013) where the authors also provided a convergence rate when global consensus takes place. One drawback of the cut-balance assumption is that it is a global assumption which may be hard to verify when not ensured by design.

A direction to search for a local assumption is to split the agents into clusters. It is reported in the literature that large scale networks often consist of sparsely interconnected clusters of densely coupled agents (Bıyık & Arcak, 2007; Chow & Kokotović, 1985; Morărescu, Martin, Girard, & Muller-Gueudin, 2016). Different algorithms have been developed to detect the clusters in such networks (Blondel, Guillaume, Lambiotte, & Lefebvre, 2008; Morărescu & Girard, 2011; Newman & Girvan, 2004). In the sequel we take advantage of the partition of network in clusters to state new conditions for consensus.

Consequently, the contribution of the present study is that, under stronger assumptions on the interaction graph, we provide a new assumption on reciprocity of communication which can be verified in a local manner. Therefore, we provide conditions for consensus that can be checked by performing a reduced number of operations.

Notation. The following notation will be used throughout the paper. The set of nonnegative integers, real and nonnegative real numbers is denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$, respectively. A non trivial subset S of a set C, denoted as $S \sqsubset C$, is a non-empty set with $S \subsetneq C$.

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2. Problem formulation

Let $\mathcal{N} \triangleq \{1, \dots, n\}$ be a set of n agents. By abuse of notation we denote both the agent and its index by the same symbol $i \in \mathcal{N}$. Each agent is characterized by a scalar state $x_i \in \mathbb{R}$, $\forall i \in \mathcal{N}$ that evolves according to the following model

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}(t)(x_j(t) - x_i(t)), \quad \forall i \in \mathcal{N} \tag{1}$$

where $a_{ij}(t) \geq 0$ are measurable functions of time representing the *communication weights/interaction strength*. Let $x(t) = (x_1(t), \dots, x_n(t))^\top \in \mathbb{R}^n$ be the overall state of the network collecting the states of all the agents. It is noteworthy that $x(t),\ t \in \mathbb{R}^+$ is uniquely defined by an initial state $x(0)$ and dynamics (1). Indeed, there exists a unique differentiable function of time $x : \mathbb{R}^+ \to \mathbb{R}^n$ whose components satisfy Eq. (1) for all $t \in \mathbb{R}^+$. We call it the *trajectory* of the overall system. We say the trajectory asymptotically reaches a consensus when there exists a common agreement value $\alpha \in \mathbb{R}$ such that

$$\lim_{t \to +\infty} x_i(t) = \alpha, \quad \forall i \in \mathcal{N}.$$

In the sequel, agents are assumed to be partitioned in m non-empty clusters: $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m \subset \mathcal{N}$, that are assumed to be given or can be easily identified. For instance clusters may correspond to groups of agents which are spatially close while different clusters are spatially distant. Let us introduce the following supplementary notation: $\mathcal{M} \triangleq \{1, \dots, m\}$ and n_i denotes the cardinality of cluster $\mathcal{C}_i$. Without loss of generality, we permute the agents' labels according to the cluster partition so that when $j \in \mathcal{C}_i$ and $j' \in \mathcal{C}_{i+1}$, $j < j'$.

Definition 1. A **directed path of length** p in a given directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{F})$ is a union of directed edges $\bigcup_{k=1}^{p}(i_k, j_k)$ such that $i_{k+1} = j_k$, $\forall k \in \{1, \dots, p-1\}$. The node j is **connected** with node i in a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{F})$ if there exists at least a directed path in $\mathcal{G}$ from i to j (i.e. $i_1 = i$ and $j_p = j$).

For two subsets of nodes $A, B \subset \mathcal{N}$, the sum of communication weights from B to A is denoted as

$$w_{A \leftarrow B}(t) = \sum_{i \in A, j \in B} a_{ij}(t).$$

The cut-balance assumption in Hendrickx and Tsitsiklis (2013) can be formulated as follows.

Hypothesis 1. There exists a constant $K \geq 1$ such that for all non trivial subsets $S \sqsubset \mathcal{N}$

$$w_{S \leftarrow (\mathcal{N} \setminus S)}(t) \leq K \cdot w_{(\mathcal{N} \setminus S) \leftarrow S}(t), \quad \forall t \geq 0. \tag{2}$$

This basically means that if a group of agents influences the remaining ones, the former group is also influenced by the remaining ones by at least a proportional amount. A comparison between the cut-balance condition and other types of communication such as existence of a spanning tree has been carried out in Hendrickx and Tsitsiklis (2013) and Martin and Girard (2013). Let us recall here the first part of Theorem 1 in Hendrickx and Tsitsiklis (2013). First, we define the graph of persistent communication.

Definition 2. A persistent edge associated with system (1) is a couple $(j, i) \in \mathcal{N} \times \mathcal{N}$ such that $\int_0^\infty a_{ij}(t)dt = +\infty$. The graph of persistent communication associated to system (1) is the graph $G = (\mathcal{N}, \mathcal{E})$ gathering all agents and including only the persistent edges, *i.e.*,

$$\mathcal{E} = \left\{ (j, i) \in \mathcal{N} \times \mathcal{N} \mid \int_0^\infty a_{ij}(t)dt = +\infty \right\}.$$

Theorem 1. *Suppose that Hypothesis 1 is satisfied for all time $t \geq 0$. Then, the trajectory of system (1) converges. Then, there is a directed path from i to j in the graph of persistent communication G if and only if there is also a directed path from j to i, and there holds in that case $\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} x_j(t)$.*

Notice that Hypothesis 1 is a global assumption which may be hard to verify when not ensured by design. The objective of this work is to propose new assumptions that can be verified locally and provides similar guarantees for the particular case of graphs partitioned in clusters. Let us introduce here the main hypotheses of this work.

Assumption 1 (*Intra-Cluster Reciprocity*)**.** There exists a constant $K_I \geq 1$ such that for any cluster $k \in \mathcal{M}$ and for all non trivial subsets $S \sqsubset \mathcal{C}_k$,

$$w_{S \leftarrow (\mathcal{C}_k \setminus S)}(t) \leq K_I \cdot w_{(\mathcal{C}_k \setminus S) \leftarrow S}(t), \quad \forall t \geq 0.$$

Assumption 2 (*Inter-Cluster Reciprocity*)**.** There exists a constant $K_E \geq 1$ such that for all non trivial subset $S \sqsubset \mathcal{M}$,

$$\sum_{k \in S} w_{\mathcal{C}_k \leftarrow (\mathcal{N} \setminus \mathcal{C}_k)}(t) \leq K_E \cdot \sum_{k \in S} w_{(\mathcal{N} \setminus \mathcal{C}_k) \leftarrow \mathcal{C}_k}(t), \quad \forall t \geq 0.$$

Assumptions 1 and 2 correspond to Hypothesis 1 within each cluster and between clusters, respectively. In Assumption 2 the equivalent cut-balance assumption is formulated in the case where each cluster is considered as a node and the communication between clusters is weighted by the sum of agent-wise communication weights. It is necessary that $K_I \geq 1$ and $K_E \geq 1$ and the equality corresponds to symmetric communications. The next assumption ensures that the total communication weight which a cluster $\mathcal{C}_k$ receives cannot exceed a proportion of the weight received by any non trivial subset of $\mathcal{C}_k$ from the rest of $\mathcal{C}_k$.

Assumption 3 (*Clustered Communication*)**.** There exists a constant $\rho > 0$ such that for each cluster $k \in \{1, \dots, m\}$ and for all non trivial subsets $S \sqsubset \mathcal{C}_k$,

$$w_{\mathcal{C}_k \leftarrow (\mathcal{N} \setminus \mathcal{C}_k)}(t) \leq \rho \cdot w_{S \leftarrow (\mathcal{C}_k \setminus S)}(t), \quad \forall t \geq 0.$$

Remark 1. The purpose of Assumption 3 is to prevent cases where two subsets of a cluster are more connected to the outside than to each other. To understand the importance of Assumption 3, we have the two following facts:

- Assumption 3 is not necessarily satisfied when Assumptions 1 and 2 hold.
- Assumptions 1 and 2 without Assumption 3 are not sufficient to obtain the global cut-balance Hypothesis 1.

A counter-example illustrated by Fig. 1 allows to prove these facts: consider the 4-agent system with communications described by $a_{12}(t) = a_{21}(t) = a_{34}(t) = a_{43}(t) = 1$ and $a_{13}(t) = a_{42}(t) = t$. All these weights form persistent edges. The other weights are assumed to be uniformly 0. The only non-trivial partition in clusters satisfying Assumptions 1 and 2 is $C_1 = \{1, 2\}$ and $C_2 = \{3, 4\}$ with $K_I = 1 = K_E$. For this partition, Assumption 3 clearly fails for instance taking $S = \{1\}$. Moreover, the global cut-balance Hypothesis 1 also fails for instance taking $S := \{1, 2, 4\}$ (see Fig. 1 for an illustration). The relation (2) in Hypothesis 1 holds only if $K \geq t,\ \forall t \geq 0$ *i.e.*, $K = \infty$, which is not feasible.