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Discrete-time distributed extremum-seeking control over networks with unstable dynamics

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Abstract: In this paper, we propose a discrete-time version of extremum-seeking control (ESC) which can be implemented by several agents communicating over a network. We consider the problem where several agents each measure a local cost and must cooperate to minimize the total cost which is the sum of the local costs. To solve this problem, we combine a dynamic average consensus algorithm with a proportional integral ESC technique in discrete-time. The consensus algorithm provides each agent with an estimate of the total cost and this estimate is used by the ESC to minimize the total cost. The main contribution of this paper is to show that the distributed agents are able to minimize the total cost despite none of the agents directly measuring it. We provide a proof of the convergence of this algorithm and a simulation example to demonstrate its effectiveness.

Keywords: Extremum-seeking control, discrete-time systems, distributed optimization

1. INTRODUCTION

Distributed real-time optimization (RTO) has received much research attention since the early 2000s. Large scale systems cannot be controlled effectively by centralized controllers due to computational complexities. A distributed system is controlled by multiple controllers, called *agents*, communicating over a network. Since the agents share the computational burden, distributed techniques are preferred for large scale systems. It can be difficult for distributed controllers to achieve global objectives since each agent only has local information. Cooperative control is a form of distributed control where consensus between agents is used achieve global objectives (Bertsekas and Tsitsiklis, 1989). Cooperative optimization approaches, such as Nedić and Ozdaglar (2009) and Johansson et al. (2009), are based on a subgradient descent approach.

Extremum-seeking control (ESC) is a model-free RTO technique which only requires measurements of a cost function(Tan et al., 2010). ESC estimates the cost function gradient and optimizes the cost by a gradient descent. Initially, ESC was developed by Krstić and Wang (2000) for single agent continuous-time systems and extended to discrete-time systems by Choi et al. (2002). Recently, Guay et al. (2015) have provided a distributed version of ESC in continuous-time using a consensus algorithm to coordinate multiple agents each implementing ESC.

In a distributed setting, agents communicate over a network to share information and reach a consensus. Consensus algorithms can be static where the constant signals are averaged (Olfati-Saber et al., 2007) or dynamic where the signals are time-varying Freeman et al. (2006). Discrete-time dynamic consensus algorithms have been proposed by Zhu and Martínez (2010) and Kia et al. (2014).

This paper provides a method for discrete-time distributed ESC. Agents measure local costs and implement a consensus algorithm from Kia et al. (2014) over a network. Each agent uses a proportional-integral (PI) ESC based on the controller from Guay (2014) using its average cost estimate. The resulting technique can solve distributed RTO problems for systems with unknown, unstable, nonlinear dynamics in discrete time.

This paper is organized as follows. We state the problem and assumptions in Section 2. In Section 3, we propose a distributed ESC to solve this problem. The convergence of this controller is proven in Section 4. A simulation example is provided in Section 5. We conclude in Section 6.

2. PROBLEM DESCRIPTION

Consider a discrete-time nonlinear control-affine system representing a sampling of a continuous time system:

$$\boldsymbol{x}[k+1] = \boldsymbol{x}[k] + \boldsymbol{f}\left(\boldsymbol{x}[k]\right) + \boldsymbol{G}\left(\boldsymbol{x}[k]\right)\boldsymbol{u}[k] \qquad (1)$$

where $k \in \mathbb{Z}$ is the time step, $\boldsymbol{x} \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathbb{U} \subseteq \mathbb{R}^m$ is the input, and $\mathbb{X}, \mathbb{U}$ are connected and compact. The vector fields, $\boldsymbol{f}$ and $\boldsymbol{G}$, are assumed to be smooth. Since this model represents a sampled-data system, $\boldsymbol{f}$ and $\boldsymbol{G}$ are obtained by the exponential map of continuous time vector fields, so they are $\mathcal{O}\left(\Delta t\right)$.

This system will be controlled by p agents which each measure a local output, y_i, and control m_i inputs, $\boldsymbol{u}_i$. The cost function can be modeled by

$$\boldsymbol{y}[k] = \boldsymbol{h}\left(\boldsymbol{x}[k]\right) \qquad (2)$$

$$J[k] = \sum_{i=1}^{p} y_i[k] = H\left(\boldsymbol{x}[k]\right) \qquad (3)$$

where $\boldsymbol{y} = [y_1, \ldots, y_p]^\top \in \mathbb{R}^p$ is the local costs and J is the total cost. We will also consider the average cost, $\frac{1}{p}J$.

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Since controllers cannot measure $\boldsymbol{x}$, an input-output model is also useful. Using the mean-value theorem (MVT), the average cost dynamics can be exactly parametrized as

$$\frac{1}{p}\Delta J[k+1] = \theta_{0,i}[k] + \boldsymbol{\theta}_{1,i}^{\top}[k]\left(\boldsymbol{u}_i[k] - \widehat{\boldsymbol{u}}_i[k]\right) = \boldsymbol{\theta}_i^{\top}[k]\boldsymbol{\phi}_i[k] \tag{4}$$

where $\boldsymbol{\theta}_i = (\theta_{0,i}, \boldsymbol{\theta}_{1,i})$, $\boldsymbol{\phi}_i = (1, \boldsymbol{u}_i - \widehat{\boldsymbol{u}}_i)$, and $\widehat{\boldsymbol{u}}_i$ is a nominal input value. $\theta_{0,i}$ is the effect of drift and all agents $j \neq i$. $\theta_{1,i}$ is the gradient of the cost with respect to $\boldsymbol{u}_i$. Using the MVT, these parameters are explicitly

$$\begin{aligned}\theta_{0,i}[k] = &\frac{1}{p}\Big(H\Big(\boldsymbol{x}[k] + \boldsymbol{f}\left(\boldsymbol{x}[k]\right) + \boldsymbol{G}\left(\boldsymbol{x}[k]\right)\widehat{\boldsymbol{u}}[k]\Big)\\ &- \frac{1}{p}H\left(\boldsymbol{x}[k]\right)\\ &+ \frac{1}{p}\sum_{j\neq i}\nabla H\left(\overline{\boldsymbol{x}}\right)\boldsymbol{G}_j\left(\boldsymbol{x}[k]\right)\left(\boldsymbol{u}_j[k] - \widehat{\boldsymbol{u}}_j[k]\right)\end{aligned} \tag{5}$$

$$\boldsymbol{\theta}_{1,i}[k] = \frac{1}{p}\boldsymbol{G}_i^{\top}\left(\boldsymbol{x}[k]\right)\nabla^{\top}H\left(\overline{\boldsymbol{x}}\right) \tag{6}$$

where $\overline{\boldsymbol{x}} = [\overline{x}_1, \ldots, \overline{x}_n]^{\top}$ with each $\overline{x}_i = f_i\left(\boldsymbol{x}[k]\right) + \delta_i \boldsymbol{g}_i[k]\left(\boldsymbol{u}[k] - \widehat{\boldsymbol{u}}[k]\right)$ for some $\delta_i \in [0,1]$.

The control objective is to stabilize the system and minimize the steady-state cost. We assume that for every $\boldsymbol{u} \in \mathbb{U}$, there exists a unique $\boldsymbol{x} \in \mathbb{X}$ such that $\boldsymbol{0} = \boldsymbol{f}\left(\boldsymbol{x}\right) + \boldsymbol{G}\left(\boldsymbol{x}\right)\boldsymbol{u}$. Therefore, we can describe the steady-state by a map, $\boldsymbol{\pi} : \mathbb{U} \to \mathbb{X}$. The steady state cost, $\ell : \mathbb{U} \to \mathbb{R}$, is defined by $\ell\left(\boldsymbol{u}\right) = H\left(\boldsymbol{\pi}\left(\boldsymbol{x}\right)\right)$. The objective is to find $\boldsymbol{u}^* = \operatorname{argmin}\left(\ell\right)$ and stabilize $\boldsymbol{x}$ to $\boldsymbol{x}^* = \boldsymbol{\pi}\left(\boldsymbol{u}^*\right)$.

For the optimization problem to be well-defined, we assume that the steady-state cost is locally convex.

Assumption 1. (Convexity of total cost). The equilibrium input-output map is such that:

$$\left.\frac{\partial \ell}{\partial \boldsymbol{u}}\right|_{\boldsymbol{u}}\left(\boldsymbol{u} - \boldsymbol{u}^*\right) \geq \beta_1 \left\|\boldsymbol{u} - \boldsymbol{u}^*\right\|^2$$

for some positive constant, $\beta_1 \in \mathbb{R}_{>0}$ and for all $\boldsymbol{u} \in \mathbb{U}$.

Note that Assumption 1 only requires that the total cost, J, is convex with respect to each input. Since local costs, y_i, are not convex, local agents cannot minimize the y_i's independently and must instead cooperate to minimize J.

The system must be stabilizable using output feedback. We assume that the system's dynamics have relative degree one so there exists a diffeomorphism, $\boldsymbol{\psi} : \mathbb{X} \to \mathbb{R}^m \times \mathbb{R}^{n_z}$ with $\boldsymbol{\psi}(\boldsymbol{x}) = (\boldsymbol{\xi}, \boldsymbol{z})$ such that

$$\begin{aligned}\boldsymbol{\xi}[k+1] &= \boldsymbol{\xi}[k] + \boldsymbol{f}_{\xi}\left(\boldsymbol{\xi}[k], \boldsymbol{z}[k]\right) + \boldsymbol{G}_{\xi}\left(\boldsymbol{\xi}[k], \boldsymbol{z}[k]\right)\boldsymbol{u}[k]\\ \boldsymbol{z}[k+1] &= \boldsymbol{z}[k] + \boldsymbol{f}_z\left(\boldsymbol{\xi}[k], \boldsymbol{z}[k]\right)\\ \boldsymbol{y}[k] &= \boldsymbol{h}_{\xi}\left(\boldsymbol{\xi}[k]\right)\end{aligned}$$

where $\boldsymbol{z} \in \mathbb{R}^{n_z}$ are the zero dynamic states and $\boldsymbol{G}_{\xi} \in \mathbb{R}^{m\times m}$ is invertible. We assume the zero dynamics are stable and the system can be stabilized by $\boldsymbol{u}_i[k] = -K_g^*\boldsymbol{\theta}_{1,i}[k] + \widehat{\boldsymbol{u}}_i$. For a stable system $K_g^* = 0$. This assumption can also be expressed in terms of a Lyapunov function.

Assumption 2. (Stabilizability). For a fixed $\widehat{\boldsymbol{u}} \in \mathbb{U}$, let $\widetilde{\boldsymbol{x}}[k] = \boldsymbol{x}[k] - \boldsymbol{\pi}\left(\widehat{\boldsymbol{u}}\right)$. Then there exists a positive definite function V_z such that:

$$\beta_2 \left\|\widetilde{\boldsymbol{x}}[k]\right\|^2 \leq V_z\left(\boldsymbol{z}[k]\right) + H\left(\boldsymbol{x}[k]\right) \leq \beta_3 \left\|\widetilde{\boldsymbol{x}}[k]\right\|^2$$

for some $\beta_2, \beta_3 \in \mathbb{R}_{>0}$. Furthermore, there exists $K_g^* \in \mathbb{R}_{\geq 0}$ and $\beta_4 \in \mathbb{R}_{>0}$such that:

$$\begin{aligned}&V_z\left(\boldsymbol{z}[k+1]\right) + H\big(\boldsymbol{x}[k] + \boldsymbol{f}(\boldsymbol{x}[k]) + \boldsymbol{G}(\boldsymbol{x}[k]) \times \ldots\\ &\big(\widehat{\boldsymbol{u}}[k] - K_g^*\boldsymbol{\theta}_1[k]\big)\big) - V_z\left(\boldsymbol{z}[k]\right) - H\left(\boldsymbol{x}[k]\right) \leq -\beta_4 \left\|\widetilde{\boldsymbol{x}}[k]\right\|^2\end{aligned}$$

Alternatively, it is sufficient to be able to bound this function by a class $\mathcal{K}_{\infty}$ function.

Note that Assumptions 1 and 2 are global assumptions on $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \in \mathbb{R}^m$. However, since $\mathbb{X}$ and $\mathbb{U}$ are compact subsets, these assumptions are equivalent to local convexity and stabilizability assumptions on $\mathbb{R}^n$ and $\mathbb{R}^m$.

To keep the problem general, the only assumptions about $\boldsymbol{f}$, $\boldsymbol{G}$, H, and $\boldsymbol{\pi}$ are Lipschitzness.

Assumption 3. (Lipschitz continuity). There exist positive constants $L_f, L_G, L_H, L_{\pi} \in \mathbb{R}_{\geq 0}$ such that:

$$\begin{aligned}\left\|\boldsymbol{f}\left(\boldsymbol{x}_1\right) - \boldsymbol{f}\left(\boldsymbol{x}_2\right)\right\| &\leq L_f \left\|\boldsymbol{x}_1 - \boldsymbol{x}_2\right\|\\ \left\|\boldsymbol{G}\left(\boldsymbol{x}_1\right) - \boldsymbol{G}\left(\boldsymbol{x}_2\right)\right\| &\leq L_G \left\|\boldsymbol{x}_1 - \boldsymbol{x}_2\right\|\\ \left\|H\left(\boldsymbol{x}_1\right) - H\left(\boldsymbol{x}_2\right)\right\| &\leq L_H \left\|\boldsymbol{x}_1 - \boldsymbol{x}_2\right\|\\ \left\|\boldsymbol{\pi}\left(\boldsymbol{u}_1\right) - \boldsymbol{\pi}\left(\boldsymbol{u}_2\right)\right\| &\leq L_{\pi} \left\|\boldsymbol{u}_1 - \boldsymbol{u}_2\right\|\end{aligned}$$

for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{X}$ and $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{U}$. Since $\boldsymbol{f}, \boldsymbol{G} \in \mathcal{O}\left(\Delta t\right)$, we also have that $L_f, L_G \in \mathcal{O}\left(\Delta t\right)$.

3. DISTRIBUTED EXTREMUM SEEKING CONTROLLER

We will use distributed ESC to minimize the steady-state cost. There are three main components to this controller: dynamic average consensus, parameter estimation, and the PI ESC (Fig. 1). The PI ESC consists of proportional and integral components and a dither signal.

ESC is based on a gradient descent in the direction of $\boldsymbol{\theta}_{1,i}$. While $\boldsymbol{\theta}_{1,i}$ is time-varying and unmeasured, it can be estimated from data of $\frac{1}{p}J$ and $\boldsymbol{u}_i - \widehat{\boldsymbol{u}}_i$ using recursive least squares (RLS) based on (4). Furthermore, since agents do not measure $\frac{1}{p}J$, a dynamic average consensus is used to provide each agent with an estimate, $\widehat{J}_i$, of the average cost which is then used in the regression.

3.1 Dynamic average consensus

The communication network can be described by a graph with agents as nodes and edges indicating which agents communicate. The Laplacian matrix, $\boldsymbol{L}$, of this graph can then describe the structure of this network. We use the consensus algorithm from Kia et al. (2014):

$$\begin{aligned}\begin{bmatrix}\widehat{\boldsymbol{J}}[k+1] - \widehat{\boldsymbol{J}}[k]\\ \boldsymbol{\rho}[k+1] - \boldsymbol{\rho}[k]\end{bmatrix} &= \begin{bmatrix}-\kappa_P\boldsymbol{I} - \kappa_I\boldsymbol{L} & -\boldsymbol{I}\\ \kappa_P\kappa_I\boldsymbol{L} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\widehat{\boldsymbol{J}}[k]\\ \boldsymbol{\rho}[k]\end{bmatrix}\Delta t\\ &+ \begin{bmatrix}\kappa_P\boldsymbol{I}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{y}[k]\Delta t + \begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{0}\end{bmatrix}\Delta\boldsymbol{y}[k]\end{aligned} \tag{7}$$

where κ_P and κ_I are tuning parameters and $\boldsymbol{\rho}$ is a vector of integrator variables. Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ denote the eigenvalues of $\boldsymbol{L}$. An assumption about the properties of $\boldsymbol{L}$ can guarantee the convergence of the consensus.

Assumption 4. (Graph properties). The network structure is a connected undirected graph. Since the graph is undirected, $\boldsymbol{L} = \boldsymbol{L}^{\top}$. Since the graph is connected, $\lambda_2 \neq 0$.