# Integral-type approximate controllability of linear parabolic integro-differential equations 

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## A R T I C L E I N F O

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#### Abstract

This paper addresses the study of the integral-type approximate controllability of linear parabolic integrodifferential equations. This new controllability is defined by imposing some additional integral-type constraints on the usual approximate controllability and therefore, can be used to keep the state close to the target. The paper is concerned with a special choice of integral kernels, which are multiples of the same exponential function. We reduce the problem of new controllability to the obtention of a unique continuation property for the suitable adjoint system.


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## 1. Introduction and main results

Let $\Omega \subset \mathbb{R}^{N}(N \geq 1)$ be a bounded domain with smooth boundary $\partial \Omega$ and $\omega \subset \Omega$ be a nonempty open set. Let $T>0$ be given. We shall denote by $Q$ the cylinder $\Omega \times(0, T)$ and by $\Sigma$ the lateral boundary $\partial \Omega \times(0, T)$. In this paper, we study the following parabolic integro-differential equation:

$$
\begin{cases}y_{t}-\Delta y=\int_{0}^{t} L(t-s) y(x, s) \mathrm{d} s+\chi_{\omega} u, & (x, t) \in Q  \tag{1.1}\\ y(x, t)=0, & (x, t) \in \Sigma, \\ y(x, 0)=y_{0}(x), & x \in \Omega\end{cases}
$$

Here, the functions $y$ and $u$ are respectively the state variable and the control variable, $L \in L^{2}(0, T)$ is the integral kernel, and $\chi_{\omega}$ denotes the characteristic function of the set $\omega$. The Eq. (1.1), as an important class of diffusion equations, can describe the physical problems in many fields such as population dynamics and heat conduction in materials with memory (see [1-4]).

Assume that $y_{0} \in L^{2}(\Omega)$ and $u \in L^{2}(\omega \times(0, T))$. By Galerkin method (see [5]), we can obtain that the system (1.1) admits a unique solution $y \in C\left([0, T] ; L^{2}(\Omega)\right) \cap L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)$. Moreover, there exists a constant $C>0$ such that
$\max _{t \in[0, T]}\|y(\cdot, t)\|_{L^{2}(\Omega)} \leq C\left(\left\|y_{0}\right\|_{L^{2}(\Omega)}+\|u\|_{L^{2}(\omega \times(0, T))}\right)$.
Our main interest concerns the approximate controllability of the linear system (1.1). The goal is to prove the existence of a control $u$ which steers the state variable and the integral term to the neighborhood of two given final configurations at the time $T$,

[^0]respectively. More precisely, the system (1.1) is said to be approximately controllable at time $T$ if, for any $y_{0} \in L^{2}(\Omega), y_{1} \in L^{2}(\Omega)$ and $\varepsilon>0$, there exists $u \in L^{2}(\omega \times(0, T))$ such that the solution $y$ of (1.1) satisfies
$\left\|y(\cdot, T)-y_{1}\right\|_{L^{2}(\Omega)} \leq \varepsilon$.
The system (1.1) is said to be integral-type approximately controllable (with the integral kernel $\widetilde{L}(\cdot) \in L^{2}(0, T)$ ) at time $T$ if, for any $y_{0} \in L^{2}(\Omega),\left(y_{1}, y_{2}\right) \in L^{2}(\Omega) \times L^{2}(\Omega)$ and $\varepsilon>0$, there exists $u \in L^{2}(\omega \times(0, T))$ such that the solution $y$ of (1.1) satisfies
$\left\|y(\cdot, T)-y_{1}\right\|_{L^{2}(\Omega)} \leq \varepsilon, \quad\left\|\int_{0}^{T} \widetilde{L}(T-s) y(\cdot, s) \mathrm{d} s-y_{2}\right\|_{L^{2}(\Omega)} \leq \varepsilon$.
Therefore, the integral-type approximate controllability is defined by imposing some additional integral-type constraints on the approximate controllability.

Let us recall the approximate controllability for the classical parabolic equation:
$\begin{cases}\theta_{t}(x, t)=\Delta \theta(x, t)+\chi_{\omega} u(x, t), & (x, t) \in Q, \\ \theta(x, t)=0, & (x, t) \in \Sigma, \\ \theta(x, 0)=\theta_{0}(x), & x \in \Omega .\end{cases}$
It is well known that for any given $T>0$ and non-empty open subset $\omega$ of $\Omega$, the system (1.2) is approximately controllable (see, for instance, $[6-9]$ and the rich references therein). Moreover, the state of (1.2) at time $T$ may be kept close to the special target zero by letting $u(x, t)=0$ for a.e. $(x, t) \in \Omega \times(T,+\infty)$.

Controllability problems for the parabolic integro-differential equations have been studied in recent years by many authors (see, for instance, [10-14]). When the integral kernel is locally
integrable and completely monotone, Barbu and Iannelli [15] obtained the approximate controllability of some parabolic integrodifferential equation. More recently, in [16] the authors considered the following integro-differential equation:
$y_{t}=a y+\Delta y+\int_{0}^{t} M(t-s) \Delta y(x, s) \mathrm{d} s+\chi_{\omega} u$,
where $M(\cdot)$ is of class $C^{1}$. They proved that the system is approximately controllable.

Unlike the classical parabolic equation (1.2), because of the appearance of the integral term $\int_{0}^{t} L(t-s) y(x, s) \mathrm{d} s$, the state of the system (1.1) at time $T$ might not be kept close to the target zero by letting $u(x, t)=0$ for a.e. ( $x, t) \in \Omega \times(T,+\infty)$. To guarantee this, in this paper we study the integral-type approximate controllability property of (1.1). The main results of this paper state that the integral-type approximate controllability of the system (1.1) holds for some integral kernel $\widetilde{L}(\cdot)$. To the best of our knowledge, the problem of new approximate controllability has not been addressed yet.

Without loss of generality we can assume $y_{0}=0$. We will use $C$ to denote a generic positive constant which may be different from line to line. $\langle\cdot, \cdot\rangle$ will stand for the usual scalar product. In what follows, we choose
$L(t)=b_{1} e^{a t}, \quad \widetilde{L}(t)=b_{2} e^{a t}$,
where $b_{1} \neq 0, b_{2} \neq 0$ and $a \in \mathbb{R}$. Now, we are ready to formulate the main results of this work.

Theorem 1.1. For any given $T>0$, the system (1.1) is integral-type approximately controllable (with the integral kernel $b_{2} e^{a \cdot}$ ) at time $T$.

Theorem 1.2. Let $a<0, b_{1}<0$ and $y_{1} \in H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$ be such that $-\Delta y_{1}+\frac{b_{1}}{a} y_{1}=0$, then for any $\varepsilon>0$, there exists a control $u \in L^{2}(\omega \times(0, T))$ such that the corresponding solution $y$ of (1.1) satisfies
$\left\|y(\cdot, t)-y_{1}\right\|_{L^{2}(\Omega)} \leq C \varepsilon, \quad \forall t>T$.
The rest of this paper is organized as follows. In Section 2, we present the proof of Theorem 1.1. Here, the key point is to prove that the adjoint system associated to our control problem (1.1) satisfies a unique continuation property. We give the proof of Theorem 1.2 in Section 3.

## 2. Proof of Theorem 1.1

This section is devoted to prove Theorem 1.1. First of all, we reduce the integral-type approximate controllability of the system (1.1) to a unique continuation property of the suitable adjoint system. Let us introduce the following adjoint system of (1.1):

$$
\begin{cases}\varphi_{t}+\Delta \varphi+\int_{t}^{T} L(s-t) \varphi(x, s) \mathrm{d} s &  \tag{2.1}\\ \quad+\widetilde{L}(T-t) z_{T}=0, & (x, t) \in Q \\ \varphi(x, t)=0, & (x, t) \in \Sigma \\ \varphi(x, T)=\varphi_{T}(x), & x \in \Omega\end{cases}
$$

where $\left(\varphi_{T}, z_{T}\right) \in L^{2}(\Omega) \times L^{2}(\Omega)$. We have the following result.
Lemma 2.1. The system (1.1) is integral-type approximately controllable at time $T$ if and only if $(0,0)$ is the only pair $\left(\varphi_{T}, z_{T}\right) \in L^{2}(\Omega) \times$ $L^{2}(\Omega)$ for which the solution $\varphi$ of (2.1) satisfies $\varphi=0$ in $\omega \times(0, T)$.

Proof. For any $u \in L^{2}(\omega \times(0, T))$, we define $G: L^{2}(\omega \times(0, T)) \rightarrow$ $L^{2}(\Omega) \times L^{2}(\Omega)$ as follows:
$G u=\left(y(\cdot, T), \int_{0}^{T} \widetilde{L}(T-t) y(\cdot, t) \mathrm{d} t\right)$,
where $y$ is the solution of (1.1) with $u \in L^{2}(\omega \times(0, T))$. Obviously, the linear system (1.1) is integral-type approximately controllable at time $T$ if and only if the range of $G$ is dense in $L^{2}(\Omega) \times L^{2}(\Omega)$. By the duality argument (see [17]), it is also equivalent to that $\operatorname{Ker} G^{*}=\{(0,0)\}$. Let us multiply by $\varphi$ the equation satisfied by $y$ and let us integrate over $Q$. Taking into account that $\varphi$ satisfies (2.1) and $y_{0}=0$, we obtain

$$
\begin{align*}
& \int_{0}^{T} \int_{\omega} u \varphi \mathrm{~d} x \mathrm{~d} t \\
& \quad=\iint_{Q}\left(y_{t}-\Delta y-\int_{0}^{t} L(t-s) y(x, s) \mathrm{d} s\right) \varphi(x, t) \mathrm{d} x \mathrm{~d} t  \tag{2.2}\\
& \quad=\int_{\Omega} y(x, T) \varphi_{T}(x) \mathrm{d} x+\iint_{Q} \widetilde{L}(T-t) z_{T}(x) y(x, t) \mathrm{d} x \mathrm{~d} t
\end{align*}
$$

Thus, we have that
$G^{*}\left(\varphi_{T}, z_{T}\right)=\chi_{\omega} \varphi, \quad \forall\left(\varphi_{T}, z_{T}\right) \in L^{2}(\Omega) \times L^{2}(\Omega)$.
Consequently, the system (1.1) is integral-type approximately controllable at time $T$ if and only if $\varphi=0$ in $\omega \times(0, T)$ implies $\varphi_{T}=z_{T}=0$. This completes the proof of Lemma 2.1.

Next, we show the proof of Theorem 1.1.
Proof of Theorem 1.1. First, we give the representation formula of the solution $\varphi$ of (2.1). Let $\left\{\lambda_{j}\right\}_{j=1}^{\infty}$ be the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary conditions satisfying
$0<\lambda_{1}<\lambda_{2} \leq \lambda_{3} \leq \cdots \rightarrow \infty$
and $\left\{\phi_{j}\right\}_{j=1}^{\infty}$ the corresponding eigenfunctions, constituting an orthonormal basis of $L^{2}(\Omega)$. Let $\left(\varphi_{T}, z_{T}\right) \in L^{2}(\Omega) \times L^{2}(\Omega)$ be given by $\varphi_{T}=\sum_{j=1}^{\infty} \varphi_{T, j} \phi_{j}, z_{T}=\sum_{j=1}^{\infty} z_{T, j} \phi_{j}$. For any $(x, t) \in Q$, put
$w(x, t)=\int_{t}^{T} e^{a(s-t)} \varphi(x, s) \mathrm{d} s$.
Then $w$ satisfies
$\begin{cases}w_{t t}+a w_{t}+\Delta w_{t}+a \Delta w-b_{1} w & \\ =b_{2} e^{a(T-t)} z_{T}, & (x, t) \in Q, \\ w(x, t)=0, & (x, t) \in \Sigma, \\ w(x, T)=0, w_{t}(x, T)=-\varphi_{T}(x), & x \in \Omega .\end{cases}$
We may represent the solution $w$ of (2.3) as
$w(x, t)=\sum_{j=1}^{\infty} \alpha_{j}(t) \phi_{j}(x), \quad \forall(x, t) \in Q$,
where $\alpha_{j}$ satisfies
$\left\{\begin{array}{l}\alpha_{j}^{\prime \prime}(t)+a \alpha_{j}^{\prime}(t)-\lambda_{j} \alpha_{j}^{\prime}(t)-a \lambda_{j} \alpha_{j}(t)-b_{1} \alpha_{j}(t) \\ \quad=b_{2} e^{a(T-t)} z_{T, j}, \quad t \in(0, T), \\ \alpha_{j}(T)=0, \alpha_{j}^{\prime}(T)=-\varphi_{T, j} .\end{array}\right.$
For any $j \in \mathbb{N}$, denote by $\mu_{j}^{ \pm}$the roots of the following quadratic equation:

$$
\mu^{2}-\left(\lambda_{j}-a\right) \mu-\left(a \lambda_{j}+b_{1}\right)=0
$$

For simplicity, assume that $\left(\lambda_{j}+a\right)^{2}+4 b_{1} \neq 0$. Then we have
$\mu_{j}^{ \pm}=\frac{\left(\lambda_{j}-a\right) \pm \sqrt{\left(\lambda_{j}+a\right)^{2}+4 b_{1}}}{2}$.

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