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Finite-parameter feedback control for stabilizing the complex Ginzburg–Landau equation

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ABSTRACT

In this paper, we prove the exponential stabilization of solutions for complex Ginzburg–Landau equations using finite-parameter feedback control algorithms, which employ finitely many volume elements, Fourier modes or nodal observables (controllers). We also propose a feedback control for steering solutions of the Ginzburg–Landau equation to a desired solution of the non-controlled system. In this latter problem, the feedback controller also involves the measurement of the solution to the non-controlled system.

1. Introduction

In this paper, we study the complex Ginzburg–Landau equation (CGLE), which is a mathematical model to describe near-critical instability waves such as a reaction–diffusion system near a Hopf-bifurcation. Specific applications of this equation include nonlinear waves, second-order phase transitions, superconductivity, superfluidity, Bose–Einstein condensation and liquid crystals. See [1] and the references therein for an overview of several phenomena described by the CGLE.

The general form of the CGLE is written as

$$u_t - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^p u - \gamma u = 0, \quad x \in \Omega, t > 0, \quad (1.1)$$

where u denotes the complex oscillation amplitude, and $\beta \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ are the (nonlinear) frequency and (linear) dispersion parameters, respectively. The constants λ and κ are assumed to be strictly positive. Ω is a general domain in $\mathbb{R}^n$ and $p > 0$ is the source-power index. (1.1) can be associated with Dirichlet, Neumann, periodic, or mixed boundary conditions depending on the physical situation. Note that Eq. (1.1) simultaneously generalizes the real reaction–diffusion equation and the nonlinear Schrödinger equation, which can be obtained in the limit as the parameter pairs (α, β) and (λ, κ) tend to zero, respectively.

It is well-known that if the *Benjamin–Feir–Newell* stability criteria ($\alpha\beta > -1$) fail to hold, then CGLE might possess unstable solutions such as the trivial solution and chaos might be observed. The Stoke's solution $u(x, t) \equiv \sqrt{\frac{\gamma}{\kappa}} e^{-i\frac{\beta\gamma}{\kappa}t}$ is another example of a space independent periodic solution (for $p = 2$) whose perturbations might be unstable [2]. Motivated by these observations, we want to study the stabilization problem for the CGLE. We will be interested only in the case $\gamma > 0$, since otherwise solutions already decay to zero, and there is no room for instability.

Controlling chaotic behavior has been one of the major subjects in the theory of evolution equations, and many approaches have been developed. One such approach involves using local or global interior control terms. Others involve external (boundary) controls, especially in models where it is difficult or impossible to access the medium. Using feedback type controls is a common tactic to suppress the chaotic behavior and bring solutions back to a stable state. However, non-feedback type controls (open loop control systems) are also used for steering solutions to or near a desired state. Exact, null, or approximate controllability models are some examples.

Regarding the stabilization of the unstable solutions of the Ginzburg–Landau equation, using an internal feedback has been a common technique. From this point of view, both global (space independent) and local (space dependent) controls were used. At the beginning, time-delay local feedback control mechanisms were used (see [3–6]). Then, a linear combination of spatially translated and time-delay local feedback control terms were introduced. For example [7] and [8] used this technique to stabilize the one and two dimensional Ginzburg–Landau equations with cubic nonlinearity, respectively. There are also some works which combine both local and global type feedback controls where a local control is by

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itself not sufficient to control the turbulence (see, for example [9] and [10]). However, in some studies (e.g., [11,12]), only global feedback controls were shown to be effective, too.

Contrary to the internal feedback control mechanisms mentioned above, boundary controls which are obtained by using the so called "back-stepping" methodology were also used to stabilize the solutions of Ginzburg–Landau evolutions, see for instance [13] and [14] for stabilization of the linearized one-dimensional Ginzburg–Landau equation from the boundary.

Using non-feedback type controls (i.e., open loop control systems) is another method to steer solutions of a system to a desired state preferably in small time. One such choice for the desired state is the zero state in which case one talks about the null-controllability. For example, [15] proved the null-controllability of the solutions of the Ginzburg–Landau equation both from the interior and the boundary via a non-feedback type control.

It is well-known that the Ginzburg–Landau equation has a finite dimensional asymptotic in-time behavior [16]. In other words, there is a finite number of degrees of freedom for the Ginzburg–Landau model. There has been some recent work utilizing this type of finite dimensionality for other dissipative systems to construct feedback controls that only use finitely many volume elements, finitely many Fourier modes, or finitely many nodal observables. For example, [17] studied the one dimensional cubic reaction–diffusion equation, also known as the Chafee–Infante equation. The authors presented a unified approach that can be applied to a large class of nonlinear partial differential equations including the CGLE that we study here. The study carried out in [17] was important since it pointed out to the fact that the finite-dimensional asymptotic behavior is sufficient for constructing feedback controls for most dissipative dynamical systems. See also [18] for a similar discussion of the nonlinear wave equation. Motivated by these recent works, we study the complex Ginzburg–Landau equation subject to a feedback control which uses only finitely many determining systems of the parameters mentioned above.

More precisely, we study the following feedback control problems in this work:

(1) L^2-stabilization of the one dimensional CGLE model with finitely many volume elements.
(2) Both L^2 and H^1-stabilization of the CGLE model with finitely many Fourier modes.
(3) Steering solutions of the CGLE model: (i) to any solution (ii) to exponentially decaying solutions.
(4) L^2-stabilization of the one dimensional CGLE model with finitely many nodal observables.

Remark 1.1 (*A Few Words on the Global Well-Posedness*). Our proofs are based on the multiplier technique and intrinsic properties of the feedback control. The multiplier method in our proofs can be easily justified by classical methods where one works on approximate solutions first and then passes to the limit in the energy estimates. The approximate solutions as well as the global solvability of the original models we study here can be obtained by using different techniques. One method is to use the maximal monotone operator theory where various terms in the equation are first replaced by their Yosida approximations; see [19] and the references therein. Another approach is of course using the Galerkin procedure where the infinite dimensional model is projected on a finite dimensional subspace. Most recently, some $L^p - L^q$ estimates have been proved on the corresponding evolution operator of the Ginzburg–Landau equation [20], from which one can also obtain the solvability of solutions. We will omit the details of these procedures in this paper, since the additional feedback control terms that we use here do not add any extra difficulties to the well-posedness problem. Hence, in all of our results we will simply assume the existence of a sufficiently nice solution (in time and space). Depending on the model posed, solutions will be assumed to be at L^2, H^1, or H^2 levels in space.

2. L^2-stabilization with finite volume elements

In this section, we consider the Ginzburg–Landau equation with finite volume elements feedback control on a bounded interval $(0, L)$ with homogeneous Neumann boundary conditions at both ends of the domain:

$$u_t - (\lambda + i\alpha)u_{xx} + (\kappa + i\beta)|u|^p u - \gamma u = -\mu \sum_{k=1}^{N} \bar{u}_k \chi_{J_k}(x), x \in (0, L), t > 0, \tag{2.1}$$

$$u_x(0, t) = u_x(L, t) = 0, \quad t > 0, \tag{2.2}$$

$$u(x, 0) = u_0(x), x \in (0, l), \tag{2.3}$$

where $\lambda, \kappa, \gamma > 0, \alpha, \beta \in \mathbb{R}, J_k \equiv \left[\frac{(k-1)L}{N}, \frac{kL}{N}\right), \bar{u}_k \equiv \frac{1}{|J_k|}\int_{J_k} u dx$, and χ_{J_k} is the characteristic function on J_k for $k = 1, 2, \ldots, N$. The right-hand side, which involves the local averages (observables) $\bar{u}_k$, is regarded as a feedback controller.

In what follows, we will use the following equivalent definition of $H^1(0, L)$-norm for convenience.

$$\|u\|^2_{H^1(0,L)} \equiv \frac{1}{L^2}\|u\|^2_{L^2(0,L)} + \|u_x\|^2_{L^2(0,L)}.$$

Theorem 2.1. *Let u be a sufficiently smooth solution of (2.1)–(2.3) with*

$$\frac{1}{N^2} < \min\left\{1 - \frac{4\gamma}{\mu}, \frac{4\lambda}{\mu L^2}\right\}. \tag{2.4}$$

Then

$$\|u(t)\|^2_{L^2(0,L)} \le e^{-\mu\left(\frac{1}{2} - \frac{2\gamma}{\mu} - \frac{1}{2N^2}\right)t}\|u_0\|^2_{L^2(0,L)}$$

for $t \ge 0$.

Proof. Taking the L^2-inner product of (2.1) with u we get

$$\int_0^L u_t \bar{u} dx + (\lambda + i\alpha)\int_0^L |u_x|^2 dx + (\kappa + i\beta)\int_0^L |u|^{p+2} dx - \gamma\int_0^L |u|^2 dx = -\mu\int_0^L I_h(u)\bar{u} dx, \tag{2.5}$$

where $I_h(u) \equiv \sum_{k=1}^{N} \bar{u}_k \chi_{J_k}(x)$. The feedback operator I_h is indeed an interpolant operator approximating the inclusion $H^1(0, T) \hookrightarrow L^2(0, L)$. More precisely, the following Bramble–Hilbert type inequality (see [17, Proposition 2.1]) holds true.

$$\|u - I_h(u)\|_{L^2(0,L)} \le h\|u\|_{H^1(0,L)} \tag{2.6}$$

where $h = \frac{L}{N}$ is the step size. Writing

$$I_h(u)\bar{u} = (I_h(u) - u)\bar{u} + |u|^2,$$

taking two times the real part of (2.5), and using the Cauchy–Schwartz inequality, we obtain

$$\frac{d}{dt}\int_0^L |u|^2 dx + 2\lambda\int_0^L |u_x|^2 dx + 2\kappa\int_0^L |u|^{p+2} dx - 2\gamma\int_0^L |u|^2 dx \le -\mu\int_0^L |u|^2 dx + \mu\left(\int_0^L |u|^2 dx\right)^{\frac{1}{2}}\left(\int_0^L |u - I_h(u)|^2 dx\right)^{\frac{1}{2}}. \tag{2.7}$$

Applying Young's inequality,

$$\frac{d}{dt}\int_0^L |u|^2 dx + 2\lambda\int_0^L |u_x|^2 dx + 2\kappa\int_0^L |u|^{p+2} dx - 2\gamma\int_0^L |u|^2 dx \le -\mu\int_0^L |u|^2 dx + \frac{\mu}{2}\int_0^L |u|^2 dx + \frac{\mu}{2}\|u - I_h(u)\|^2_{L^2(0,L)}. \tag{2.8}$$