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Safety verification for distributed parameter systems using barrier functionals☆

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ABSTRACT

We study the safety verification problem for a class of distributed parameter systems described by partial differential equations (PDEs), *i.e.*, the problem of checking whether the solutions of the PDE satisfy a set of constraints at a particular point in time. The proposed method is based on an extension of barrier certificates to infinite-dimensional systems. In this respect, we introduce *barrier functionals*, which are functionals of the dependent and independent variables. Given a set of initial conditions and an unsafe set, we demonstrate that if such a functional exists satisfying two (integral) inequalities, then the solutions of the system do not enter the unsafe set. Therefore, the proposed method does not require finite-dimensional approximations of the distributed parameter system. Furthermore, for PDEs with polynomial data, we solve the associated integral inequalities using semi-definite programming (SDP) based on a method that relies on a quadratic representation of the integrands of integral inequalities. The proposed method is illustrated through examples.

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1. Introduction

Many real-world engineering systems are described by partial differential equation (PDE) models, which include derivatives with respect to both space and time. For example, mechanics of fluid flows [1], dynamics of spatially inhomogeneous robot swarms [2], and the magnetic flux profile in a tokamak [3] are all described by PDEs. However, compared to systems described by ordinary differential equations (ODEs), the analysis of PDE systems is more challenging. For instance, the solutions to PDEs belong to infinite dimensional (function) spaces, where the norms are not equivalent, as opposed to Euclidean spaces for ODEs. Hence, properties such as stability [4] and input–output gains [5] may differ from one norm to another.

One interesting and unresolved problem in the analysis of PDEs is safety verification. That is, given the set of initial conditions, check whether the solutions of the PDE satisfy a set of constraints, or, in other words, whether they are safe with respect to an unsafe set. Reliable safety verification methods are fundamental for designing safety critical systems, such as life support systems [6], satellite docking systems [7] and wind turbines [8]. The safety verification problem is well-studied for ODE systems (see the survey paper [9]). Methods based on the approximation of the reachable sets are considered in [10] for linear systems and in [11] for nonlinear systems. Another method for safety verification, which does not require the approximation of reachable sets, uses barrier certificates. Barrier certificates [12] were introduced for model invalidation of ODEs with polynomial vector fields and have been used to address safety verification of nonlinear and hybrid systems [13] and safety analysis of time-delay systems [14]. Exponential barrier functions were proposed in [15] for finite-time regional verification of stochastic nonlinear systems. Moreover, compositional barrier certificates and converse results were studied in [16] and [17,18], respectively.

The application of barrier certificates goes beyond just analysis. Inspired by the notion of control Lyapunov functions [19] and Sontag's formula [20], Wieland and Allgöwer [21] introduced control barrier functions (CBFs) and formulated a controller synthesis method that ensures safety with respect to an unsafe set. This has sparked several subsequent studies on control barrier functions [22,23].

In this paper, we study the safety verification problem for PDEs using barrier certificates. The proposed method employs a functional of the dependent and independent variables called the

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barrier functional. We show that the safety verification problem can be cast as the existence of a barrier functional satisfying a set of integral inequalities. For PDEs with polynomial data, we demonstrate that the associated integral inequalities can be solved using semi-definite programming (SDP) based on the results in [24], which were also used in [5] to solve dissipation inequalities for PDEs and in [25] for input–output analysis of fluid flows. In this respect, we formulate an S-procedure-like scheme for checking integral inequalities subject to a set of integral constraints. The proposed method is illustrated by two examples.

A preliminary application of the proposed method to bounding nonlinear output functionals of nonlinear time-dependent PDEs was discussed in [26]. In this regard, a scheme for bounding linear output functionals of linear stationary PDEs using SDPs was presented in [27] based on moment relaxation techniques. In addition, a moment-relaxation-based method was formulated in [28] to find smooth approximations of the solutions to nonlinear stationary PDEs using a finite-difference discretization of the domain and maximum entropy estimation.

This paper is organized as follows. In the next section, we present some preliminary definitions. In Section 3, we describe a method based on barrier functionals for safety verification of PDEs. In Section 4, we discuss the computational formulation of the barrier functionals method and describe a scheme for verifying integral inequalities subject to integral constraints. We illustrate the proposed results using two examples in Section 5 and conclude the paper in Section 6.

Notation: The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$ and the set of nonnegative reals by $\mathbb{R}_{\geq 0}$. The n-dimensional set of positive integers is denoted by $\mathbb{N}^n$, and the n-dimensional space of non-negative integers is denoted by $\mathbb{N}^n_{\geq 0}$. We use M' to denote the transpose of matrix M. The set of real symmetric matrices is denoted $\mathbb{S}^n = \{A \in \mathbb{R}^{n\times n} \mid A = A'\}$. The ring of polynomials on a real variable x is denoted $\mathcal{R}[x]$, and, for $f \in \mathcal{R}[x]$, $deg(f)$ denotes the degree of f in x. A domain Ω is an open subset of $\mathbb{R}^n$ and the boundary of Ω is denoted $\partial\Omega$. The space of k-times continuous differentiable functions defined on Ω is denoted by $\mathcal{C}^k(\Omega)$ and the space of $\mathcal{C}^k(\Omega)$ functions mapping to a set Γ is denoted $\mathcal{C}^k(\Omega;\Gamma)$. For a multivariable function $f(x,y)$, we use $f(x,\cdot) \in \mathcal{C}^k[x]$ to denote the k-times continuous differentiability of f with respect to variable x. If $p \in \mathcal{C}^1(\Omega)$, then $\partial_x p$ denotes the derivative of p with respect to variable $x \in \Omega$. In addition, we adopt Schwartz's multi-index notation. For $u \in \mathcal{C}^\alpha(\Omega;\mathbb{R}^m)$, $\Omega \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_{\geq 0}$, defining matrix $A \in \mathbb{N}^{\sigma(m,\alpha)\times n}_{\geq 0}$, $\sigma(n,\alpha) = \frac{(n+\alpha)!}{n!\alpha!}$ (denote its ith row A_i) which contains a set of ordered elements satisfying $\Sigma_j A_{ij} \leq \alpha$, we have

$$D^\alpha u := \left(u_1, \partial_x u_1, \ldots, \partial_x^{A_\sigma} u_1, \ldots, u_m, \partial_x u_m, \ldots, \partial_x^{A_\sigma} u_m\right),$$

where $\partial_x^{A_i}(\cdot) = \partial_x^{A_{i1}}(\cdot)\cdots\partial_x^{A_{in}}(\cdot)$. We use the same multi-index notation to denote a vector of monomials up to degree α on a variable x as $\eta^\alpha(x)$. For instance, for $x \in \mathbb{R}^2$, $\eta^2(x) = (1, x_1, x_2, x_1^2, x_1x_2, x_2^2)$. The Hilbert space of functions defined over the domain Ω with the norm $\|u\|_{\mathcal{W}^p_\Omega} = \left(\int_\Omega \sum_{i=0}^p (\partial_{x_i} u)'(\partial_{x_i} u)\ dx\right)^{\frac{1}{2}}$ is denoted $\mathcal{W}^p_\Omega$. By $f \in \mathcal{L}^2(\Omega;\Gamma)$, we denote a square integrable function mapping $\Omega \subseteq \mathbb{R}^n$ to $\Gamma \subseteq \mathbb{R}^m$. Also, for an operator $\mathscr{A}$, $Dom(\mathscr{A})$ and $Ran(\mathscr{A})$ denote its domain and range, respectively. The notation $\lceil\cdot\rceil$ denotes the ceiling function.

2. Preliminaries

In this section, we present some definitions and preliminary results. We study a class of forward-in-time PDE systems. Let $\mathcal{U}$ be a Hilbert space. Consider the following differential equations

$$\begin{cases} \partial_t u(t,x) = \mathscr{F}u(t,x), \quad x \in \Omega \subset \mathbb{R}^n,\ t \in [0,T], \\ y(t) = \mathscr{H}u(t,x) \\ u(0,x) = u_0(x) \in \mathcal{U}_0 \subset Dom(\mathscr{F}) \\ u \in \mathcal{U}_b \end{cases} \tag{1}$$

where $\mathcal{U}_b$ is a subspace of $\mathcal{U}$, the state-space of system (1), defined by the boundary conditions, $\mathscr{H} : \mathcal{U} \to \mathbb{R}$ and $Dom(\mathscr{H}) \supseteq \mathcal{U}$. It is assumed that (1) is well-posed. Appendix A reviews some aspects of the well-posedness of PDEs. While these results are important, studying the well-posedness of system (1) is beyond the scope of the current paper.

We call the set

$$\mathcal{Y}_u = \left\{u \in \mathcal{U} \mid \mathscr{H}u \leq 0\right\},$$

the *unsafe set*.

Consider the following properties of trajectories related to an initial set $\mathcal{U}_0$ and an unsafe set $\mathcal{Y}_u$.

Definition 2.1 (*Safety at Time T*)**.** Let $u \in \mathcal{U}$. For a set $\mathcal{U}_0 \subseteq \mathcal{U}$, an unsafe set $\mathcal{Y}_u$, satisfying $\mathcal{U}_0 \cap \mathcal{Y}_u = \emptyset$, and a positive scalar T, system (1) is *$\mathcal{Y}_u$-safe at time T* , if the solutions $u(t,x)$ of system (1) satisfy $y(T) \notin \mathcal{Y}_u$ for all $u(0,x) \in \mathcal{U}_0$.

Definition 2.2 (*Safety*)**.** System (1) is *$\mathcal{Y}_u$-safe*, if it is safe with respect to $\mathcal{Y}_u$ in the sense of Definition 2.1 for all $T > 0$.

We are interested in solving the following problem:

Problem 2.3. Given sets $\mathcal{Y}_u, \mathcal{U}_0$ and a constant $T > 0$, verify that system (1) is $\mathcal{Y}_u$-safe at time T.

To this end, we define a time-dependent functional of the states of the PDE and time

$$B(t,u) = \mathscr{B}(t)u, \tag{2}$$

where $\mathscr{B} : Dom(\mathscr{B}) \to \mathbb{R}$. We refer to this functional as the barrier functional. Note that this extension of barrier certificates [12] enables us to address sets that are defined on infinite-dimensional spaces. In the subsequent section, we show that the barrier functional provides the means to characterize a barrier between the set of initial conditions and the unsafe set.

3. Barrier functionals for safety verification of PDEs

In this section, we present conditions to obtain certificates that trajectories starting in the set $\mathcal{U}_0$ are $\mathcal{Y}_u$-safe at a particular time instant T. Such a formulation also allows obtaining performance estimates whenever the unsafe set represents a performance index.

Next, we provide a solution to Problem 2.3 based on the construction of barrier functionals satisfying a set of inequalities.

Theorem 3.1 (*Safety Verification for Forward PDE Systems*)**.** *Consider the PDE system described by (1). Let $u \in \mathcal{U}_b$. Given a set of initial conditions $\mathcal{U}_0 \subseteq \mathcal{U}_b$, an unsafe set $\mathcal{Y}_u$, such that $\mathcal{U}_0 \cap \mathcal{Y}_u = \emptyset$, and a constant $T > 0$, if there exists a barrier functional $B(t,u(t,x)) \in \mathcal{C}^1[t]$ as in (2), such that the following inequalities hold*

$$B(T,u(T,x)) - B(0,u_0(x)) > 0, \quad \forall u(T,x) \in \mathcal{Y}_u,\ \forall u_0 \in \mathcal{U}_0, \tag{3a}$$

$$\frac{dB(t,u(t,x))}{dt} \leq 0, \quad \forall t \in [0,T],\ \forall u \in \mathcal{U}_b, \tag{3b}$$

where $\frac{d(\cdot)}{dt}$ denotes the total derivative, along the solutions of (1), then the solutions of (1) are $\mathcal{Y}_u$-safe at time T (cf. Definition 2.1).

Proof. The proof is by contradiction. Assume there exists a solution of (1) such that, at time T, $u(T,x) \in \mathcal{Y}_u$ and inequality (3a) holds. From (3b), it follows that

$$\frac{dB(t,u(t,x))}{dt} \leq 0, \tag{4}$$

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