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A topological obstruction in a control problem

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ABSTRACT

One of the important discoveries in control theory is a topological obstruction to continuous feedback stabilization for general nonlinear control systems. In this note we describe another topological obstruction arising from a very different control problem called the *reach control problem*. Motivated by a classical topological obstruction for extending continuous maps on spheres, we introduce the problem of extending continuous maps on simplices. It is shown that the same condition as in the sphere case gives rise to the obstruction.

1. Introduction

This paper regards the Reach Control Problem (RCP). The RCP seeks to find a feedback control which drives the trajectories of an affine system initialized in a simplex to reach and exit a facet of the simplex. We direct the reader to [1,2] and other cited works which regard the RCP for a more substantial discussion. For the purposes of this introduction, it suffices to state that this line of research is motivated by the desire to satisfy complex control specifications given in a constrained state space; this research thrust is also present, for instance, in [3–5], although the settings and methods of those papers are vastly different from the reach control setting.

The classical theory of controllability is largely focused on control strategies in the Euclidean space, and the methods for dealing with control theory in constrained spaces are still under development. While we make no claim to provide an extensive discussion of such efforts within this paper, we direct the reader to a vast discussion and body of references in [6]. We additionally note that this work is related in spirit to results of [7–11], as well as other works which deal with controllability or reachability of systems with constraints. In the interest of space we will not delve into a further discussion of these papers. While the results in the current paper are motivated by similar lines of inquiry as the previously mentioned works, the theoretical results obtained and methods used to do so are invariably vastly different from the current paper. We point the reader to those papers and the references contained therein for more information on the control of systems with state and input constraints. In this paper, we investigate an obstruction to solving the RCP via continuous state feedback. Noteworthy is the analogy with the topological obstruction to continuous feedback stabilization [12,13].

The RCP has been extensively studied with emphasis on finding a complete class of controls to solve the problem [14–16]. Affine state feedback has played the dominant role, in analogy with linear state feedback to solve the stabilization problem for linear systems [1,2]. Under a special triangulation of the state space, it has been shown that affine feedback and continuous state feedback are equivalent with respect to solving the RCP [14]. Under the same triangulation, piecewise affine feedback or time-varying affine feedback may be used when continuous state feedbacks fail [15,16]. However, determining a set of easily verifiable sufficient and necessary conditions for the solvability of the RCP under continuous state feedback solving the RCP has thus far remained elusive. The recent research effort on the topic of a topological obstruction is an attempt to obtain easily verifiable strong necessary, albeit not sufficient, condition for the solvability of the RCP.

Our investigation of a topological obstruction has precursors in two specific areas. The initial piece of motivation is given by [16], where we investigated a situation when continuous state feedback fails under a special triangulation. It was discovered that the failure arises from two conditions. First, the control system is underactuated, meaning there are not sufficient control inputs to resolve the requirements of the RCP. Second, available control directions are not adapted to the simplex so that even with high gain control, closed-loop equilibria appear in the simplex using continuous state feedback, resulting in a failure to solve the problem. The main tool to prove existence of equilibria was Sperner's lemma [14]. We use a similar proof method here. This paper can be regarded as a generalization of [16] to the case of arbitrary triangulations.

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The second precursor of this investigation is contained in [17, Theorem 1]. It identifies a cone condition relating the available control directions to the geometry of the simplex as a necessary condition to solve the RCP by continuous state feedback. The result was for single input systems only. Due to its reliance on the intermediate value theorem, the proof cannot be easily extended to systems with multiple inputs. The same cone condition again emerges in the present work; however, now we work with multi-input systems.

The topological obstruction has already been investigated in [18–21]. We now clarify the differences between the present paper and our previous work. All three above papers use topological methods for establishing sufficient and necessary conditions for the existence of a topological obstruction to the solvability of RCP. The paper [19] treats the case of two- and three-dimensional simplices, [18] discusses systems with two inputs, and [20] and [21] assume that an underlying affine system is controllable, and use that assumption to provide the most general currently available characterization of the topological obstruction problem. In this paper, we assume a special symmetric structure on the set of possible equilibria in the simplex, which does not appear as a requirement in either of the three above papers. This structure enables us to use algebraic methods to show that the cone condition identified in [17] is a necessary and sufficient condition for the existence of a topological obstruction, under the assumptions of this paper. This condition is significantly simpler than the conditions discovered in previous papers, at the expense of more restrictive assumptions. The proof methods are also different. Specifically, [19] heavily uses retractions, while [18,20], and [21] use homotopy and extension theory. In this paper, topological methods are mostly limited to our work on the sphere, while the obstruction on a simplex is investigated using linear algebra and the Knaster–Kuratowski–Mazurkiewicz (KKM) lemma.

We also note the recent work of [22] and [23], which opens the space for connecting the problem of a topological obstruction to wider results in control theory. In [22], the problem of the solvability of the RCP is related to the notion of local controllability. However, this connection has not yet been formalized. More significantly, [23] poses the RCP in terms of a problem on positive systems. Hence, the results presented in this paper can be interpreted as results on the existence of equilibria in the setting of positive systems. This will be further briefly discussed in Section 2.

The contributions of the paper are as follows. In Section 4, we formulate the problem of a topological obstruction on the sphere as an analogue to the known question of a topological obstruction to the RCP on a simplex. Using degree theory, that problem is then solved at the end of Section 4. In Section 5, we go back to the problem of a topological obstruction on a simplex. We solve the problem under certain additional assumptions on the structure of the set of possible equilibria.

Notation. Let $\mathcal{X}$ and $\mathcal{Y}$ be sets. The direct sum of $\mathcal{X}$ and $\mathcal{Y}$ is denoted by $\mathcal{X} \oplus \mathcal{Y}$. If $\mathcal{X}$ is contained in a topological space, notation $\mathcal{X}^\circ$ denotes the (relative) interior of $\mathcal{X}$, while $\overline{\mathcal{X}}$ denotes its (relative) closure, and $\partial\mathcal{X}$ denotes its (relative) boundary. Notation $id_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ represents the identity map. The symbol for an n-dimensional unit ball is $\mathbb{B}^n$, and for an $(n-1)$-dimensional unit sphere is $\mathbb{S}^{n-1}$. Notations $\mathrm{co}\{v_1, \dots, v_k\}$ and $\mathrm{span}\{v_1, \dots, v_k\}$ denote the convex hull and vector subspace generated by points $v_1, \dots, v_k$, respectively.

2. Reach control problem

In this section we introduce the control problem which gives rise to our study of a topological obstruction. We consider an n-dimensional simplex $\mathcal{S} := \mathrm{co}\{v_0, \dots, v_n\}$, the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$. Let its vertex set be $V := \{v_0, \dots, v_n\}$ and its facets $\mathcal{F}_0, \dots, \mathcal{F}_n$. The facet will be indexed by the vertex it does not contain. Equivalently, the facet $\mathcal{F}_i$ is the convex hull of vertices $v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n$. Without loss of generality, we assume that $v_0 = 0$. Let h_j, $j \in \{0, \dots, n\}$, be the unit normal vector to each facet $\mathcal{F}_j$ pointing outside of the simplex. Facet $\mathcal{F}_0$ is called the *exit facet*. Let $I := \{1, \dots, n\}$ and define $I(x)$ to be the minimal index set among $\{0, \dots, n\}$ such that $x \in \mathrm{co}\{v_i \mid i \in I(x)\}$. For $x \in \mathcal{S}$ define the closed, convex cone

$$\mathcal{C}(x) := \left\{ y \in \mathbb{R}^n \mid h_j \cdot y \le 0,\ j \in I \setminus I(x) \right\}. \tag{2.1}$$

(Note that h_0 never appears and $\mathcal{C}(x) = \mathbb{R}^n$ for $x \in \mathcal{S}^\circ$.) Fig. 1, modified from [24], illustrates our notation for a 2D simplex.

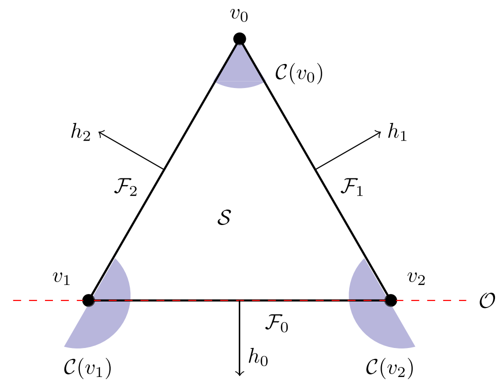

Fig. 1. A simplex $\mathcal{S} = \mathrm{co}\{v_0, v_1, v_2\}$ with vertices $V = \{v_0, v_1, v_2\}$ and facets $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$. The unit normal vector of each $\mathcal{F}_i$ pointing out of $\mathcal{S}$ is h_i. The cones $\mathcal{C}(v_i)$ are shown attached at each v_i.

We consider the affine control system on $\mathcal{S}$:

$$\dot{x} = Ax + Bu + a\,, \qquad x \in \mathcal{S}\,, \tag{2.2}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\mathrm{rank}(B) = m$. Let $\mathcal{B} = \mathrm{Im}(B)$, the image of B. Let $\phi_u(t, x_0)$ denote the trajectory of (2.2) starting at x_0 under control input u. We are interested in studying the reachability of the exit facet $\mathcal{F}_0$ from $\mathcal{S}$.

Problem 1 (*Reach Control Problem (RCP)*). Consider system (2.2) defined on $\mathcal{S}$. Find a feedback $u(x)$ such that: for each $x_0 \in \mathcal{S}$ there exist $T \ge 0$ and $\delta > 0$ such that

- (i) $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$,
- (ii) $\phi_u(T, x_0) \in \mathcal{F}_0$, and
- (iii) $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T+\delta)$.

The RCP says that trajectories of (2.2) starting from initial conditions in $\mathcal{S}$ exit $\mathcal{S}$ through the exit facet $\mathcal{F}_0$ in finite time, while not first leaving $\mathcal{S}$. In order for a feedback $u(x)$ to solve Problem 1, $Ax + Bu + a$ cannot have any equilibria in $\mathcal{S}$. We observe that $Ax + Bu + a$ can vanish for an appropriate choice of u only if $x \in \mathcal{O}$ where $\mathcal{O} := \{ x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B}\}$. Thus, if $u(x)$ is a continuous state feedback, then equilibria of the closed-loop system can only appear in

$$\mathcal{G} := \mathcal{S} \cap \mathcal{O}\,.$$

Additionally, to solve the RCP we require conditions that disallow trajectories to exit from the facets $\mathcal{F}_i$, $i \in I$. We say that a state feedback $u(x)$ satisfies the *invariance conditions* if

$$Ax + Bu(x) + a \in \mathcal{C}(x)\,,\ x \in \mathcal{S}\,. \tag{2.3}$$

The invariance conditions are necessary conditions to solve the RCP [16].

We wish to investigate when there exists a continuous $u(x)$ satisfying (2.3) such that there are no closed-loop equilibria in $\mathcal{G}$.