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On stability of the Kalman filter for discrete time output error systems

Qinghua Zhang

INRIA-IFSTTAR, , Campus de Beaulieu, 35042 Rennes Cedex, France

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ABSTRACT

The stability of the Kalman filter is classically ensured by the uniform complete controllability regarding the process noise and the uniform complete observability of linear time varying systems. This paper studies the case of discrete time output error (OE) systems, in which the process noise is totally absent. The classical stability analysis assuming the controllability regarding the process noise is thus not applicable. It is shown in this paper that the uniform complete observability is sufficient to ensure the stability of the Kalman filter applied to time varying OE systems, regardless of the stability of the OE systems. Though the continuous time case has been studied recently, the results on continuous time systems cannot be directly transposed to discrete time systems, because of a difficulty related to the observability of the discrete time filter error dynamics system.

1. Introduction

The well known Kalman filter has been extensively studied and is being applied in many different fields [1–5]. The purpose of the present paper is to study the stability of the Kalman filter in a particular case not yet covered in the literature: the absence of process noise in the state equation of a *discrete time* linear time varying (LTV) system. Such systems are known as *output error* (OE) systems in the literature on system identification. The recent studies on *continuous time* OE systems in [6,7] have been mainly motivated by applications where state equations originate from physical laws that are believed sufficiently accurate. For *discrete time* systems considered in this paper, the motivation is mainly for OE system identification [8–10]. In control applications, the use of OE models has the advantage of focusing system identification on the dynamics of the controlled plant, rather than on noise properties [9]. The result presented in this paper ensures the stability of the Kalman filter applied to LTV OE systems. This result is particularly useful for linear parameter varying (LPV) system identification based on prediction error minimization, as it ensures stable predictions, regardless of the stability of the estimated LPV models during the iterations of prediction error minimization.

While the optimal properties of the Kalman filter are frequently recalled, its stability properties are less often mentioned in the recent literature. The classical stability analysis is based on both the uniform complete controllability *regarding the process noise* and the uniform complete observability of LTV systems [2,11]. In the case of OE systems, there is no process noise at all in the state equation, hence the controllability regarding the process noise cannot be fulfilled.

The stability of the Kalman filter for *continuous time* OE systems has been recently studied in [6,7]. It is often straightforward to transpose theoretic results from continuous time systems to discrete time systems, and vice versa, but there are exceptions. For the stability problem studied in this paper, there are two main extra difficulties for discrete time systems.

First, in the continuous time case, the observability of an OE system induces the observability of its Kalman filter error dynamics system, and this observability plays an important role in the stability analysis of the error dynamics. In the discrete time case, however, the observability of the Kalman filter error dynamics system cannot be induced in a similar way, as explained in Section 5.

Second, in the continuous time case, in the first step of the proof of the Kalman filter asymptotic stability, the Lyapunov stability is naturally proved within a few lines in [7]. In the discrete time case, however, by indirectly analyzing the Kalman filter error dynamics, due to the complexity related to the separation between the prediction step and the update step (no such separation exists in the continuous time case), the proof of the Lyapunov stability takes more than half a page (see the proof of Theorem 1 in this paper), with non trivial choices of appropriate equalities involved in the discrete time case only.

The classical *optimality* results of the Kalman filter are also valid in the case of OE systems [2, chapter 7]. However, it is necessary to complete the stability analysis, as the classical results are not applicable in this case.

As the *main contribution* of this paper, it will be shown that the uniform complete observability is sufficient to guarantee the stability of the dynamics of the Kalman filter applied to a discrete time LTV OE system, *regardless of the stability of the OE system itself*. The boundedness of the state estimate covariance, as well as the

E-mail address: qinghua.zhang@inria.fr.

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boundedness of the Kalman gain, will also be proved under the same condition. These results complete the classical results [2,11], which do not cover the case of OE systems.

Preliminary results of this study have been presented at the IFAC Word Congress [12]. The present paper enriches these preliminary results with technical details, notably the relationship between the Kalman filter error dynamics and the Kalman predictor error dynamics, and also with numerical examples.

The rest of this paper is organized as follows. Some preliminary elements are introduced in Section 2. The problem considered in this paper is formulated in Section 3. The boundedness of the Kalman filter for OE systems is analyzed in Section 4, and the asymptotic stability of the Kalman filter is established in Section 5. Numerical examples are presented in Section 6. Finally, concluding remarks are drawn in Section 7.

2. Definitions and basic facts

Let m and n be any two positive integers. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm. For a matrix $A \in \mathbb{R}^{m\times n}$, $\|A\|$ denotes the matrix norm induced by the Euclidean vector norm, which is equal to the largest singular value of A. Then $\|Ax\| \leq \|A\|\|x\|$ for all $A \in \mathbb{R}^{m\times n}$ and all $x \in \mathbb{R}^n$. For two real square symmetric positive definite matrices A and B, $A > B$ means $A - B$ is positive definite.

Let $A(k) \in \mathbb{R}^{m\times n}$ be a sequence of matrices for $k = 0, 1, 2, \ldots$. It is said (upper) bounded if $\|A(k)\|$ is bounded.

Consider the homogeneous discrete time LTV system

$$x(k) = A(k)x(k–1) \tag{1}$$

with $x(k) \in \mathbb{R}^n$ and $A(k) \in \mathbb{R}^{n\times n}$, and with the associated state transition matrix defined as

$$\Phi(k, k) = I_n \tag{2}$$

$$\Phi(k, l) = A(k)A(k–1)\cdots A(l + 1) \tag{3}$$

with I_n denoting the $n \times n$ identity matrix. Then $x(k) = \Phi(k, l)x(l)$.

Definition 1. System (1) is *Lyapunov stable* if there exists a positive constant γ such that, for all integers k, k_0 satisfying $k \geq k_0$, the following inequality holds

$$\|\Phi(k, k_0)\| \leq \gamma. \quad \Box \tag{4}$$

Definition 2. System (1) is *asymptotically stable* if it is Lyapunov stable and if the following limiting behavior holds

$$\lim_{k\to+\infty} \|x(k)\| = 0 \tag{5}$$

for any initial state $x(0) \in \mathbb{R}^n$. $\Box$

The following uniform complete observability[1] definition for LTV systems follows [11].

Definition 3. The matrix pair $\{A(k), C(k)\}$ with $A(k) \in \mathbb{R}^{n\times n}$ and $C(k) \in \mathbb{R}^{m\times n}$ is *uniformly completely observable* if there exist positive constants ρ_1, ρ_2 and a positive integer h such that, for all $k \geq h$, the following inequalities hold

$$\rho_1 I_n \leq \sum_{s=k-h}^{k} \Phi^T(s, k)C^T(s)R^{-1}(s)C(s)\Phi(s, k) \tag{6}$$

$$\leq \rho_2 I_n \tag{7}$$

with some bounded symmetric positive definite matrix $R(s) \in \mathbb{R}^{m\times m}$ (typically the covariance matrix of the output noise in a stochastic state space system). $\Box$

[1] Some variants of the definition of the uniform complete observability exist in the literature. The definition recalled here follows [11].

3. Problem formulation and assumptions

In this section the considered OE system and its Kalman filter are first formulated, before the statement of the assumptions for stability analysis.

3.1. Output error system and Kalman filter

The discrete time output error (OE) systems considered in this paper are in the form of

$$x(k) = A(k)x(k–1) + B(k)u(k) \tag{8a}$$

$$y(k) = C(k)x(k) + R^{\frac{1}{2}}(k)v(k) \tag{8b}$$

where $k = 0, 1, 2, \ldots$ represents the discrete time index, $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^l$ the input, $y(k) \in \mathbb{R}^m$ the output, $v(k) \in \mathbb{R}^m$ a white Gaussian noise with identity covariance matrix, $A(k)$, $B(k)$, $C(k)$, $R(k)$ are real matrices of appropriate sizes. The noise covariance matrix $R(k)$ is symmetric positive definite. The notation $R^{\frac{1}{2}}(k)$ denotes the symmetric positive definite matrix square root of $R(k)$. The initial state $x(0) \in \mathbb{R}^n$ is a random vector following the Gaussian distribution $x(0) \sim \mathcal{N}(x_0, P_0)$ with $x_0 \in \mathbb{R}^n$ and $P_0 \in \mathbb{R}^{n\times n}$.

Compared to the general discrete time state equation

$$x(k) = A(k)x(k - 1) + B(k)u(k) + Q^{\frac{1}{2}}(k)w(k) \tag{9}$$

with the process noise $w(k)$ and the covariance matrix $Q(k)$, an OE system corresponds to the particular case of $Q(k) \equiv 0$.

After the initialization with $P(0|0) = P_0$ and $\hat{x}(0|0) = x_0$, the Kalman filter for the OE system (8) consists of the following recursions for $k = 1, 2, \ldots,$

$$P(k|k-1) = A(k)P(k-1|k-1)A^T(k) \tag{10a}$$

$$\Sigma(k) = C(k)P(k|k–1)C^T(k) + R(k) \tag{10b}$$

$$K(k) = P(k|k–1)C^T(k)\Sigma^{-1}(k) \tag{10c}$$

$$P(k|k) = [I_n - K(k)C(k)]P(k|k–1) \tag{10d}$$

$$\hat{x}(k|k–1) = A(k)\hat{x}(k–1|k–1) + B(k)u(k) \tag{10e}$$

$$\tilde{y}(k) = y(k) - C(k)\hat{x}(k|k–1) \tag{10f}$$

$$\hat{x}(k|k) = \hat{x}(k|k–1) + K(k)\tilde{y}(k). \tag{10g}$$

For general LTV systems with a process noise as in the state equation (9), the first equation of the Kalman filter would be

$$P(k|k - 1) = A(k)P(k - 1|k - 1)A^T(k) + Q(k) \tag{11}$$

with an extra term $Q(k)$ representing the process noise covariance matrix. Therefore, the OE system Kalman filter corresponds to the particular case with $Q(k) \equiv 0$ of the general LTV system Kalman filter.

It is known that the dynamics of the general LTV system Kalman filter is stable, provided the matrix pair $\{A(k), Q^{\frac{1}{2}}(k)\}$ is uniformly completely controllable and the matrix pair $\{A(k), C(k)\}$ is uniformly completely observable [2,11]. As OE systems correspond to the case with $Q(k) \equiv 0$, the controllability condition cannot be satisfied. Consequently, the classical results on the *stability* of the Kalman filter cannot be applied here. The main purpose of the present paper is to study the Kalman filter stability in this particular case.

It will be shown that the error dynamics of the Kalman filter (10) is asymptotically stable, and that its iteratively computed variables are all bounded.

Note that the classical *optimality* results of the Kalman filter remain valid in the case of OE systems [2, chapter 7].