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Constructing a controllable graph under edge constraints

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ABSTRACT

This paper is concerned with the problem of constructing a controllable graph subject to some practical edge constraints. Specifically, suppose the total amount of vertices and the upper bounds on the graph diameter or on the vertex degree are given. We consider the problem of exploring a class of feasible graphs that satisfy the constraints. Using the hybrid of a path graph and an antiregular graph we propose a simple and systematic method to generate a class of controllable graphs whose diameters or degrees cover the full possible ranges. The method to select the control vector to ensure the controllability of the combined graph is also proposed. Numerical examples are provided to demonstrate our results.

1. Introduction

The study of dynamical systems defined on graphs is one of the most popular themes in the area of system and control engineering in the past decade. The systems are in various forms of networks that connect smaller subsystems, known as agents, to accomplish pre-specified tasks cooperatively [1,2]. For example, in engineering practice a group of robots might be used to form a multi-agent system (MAS) to monitor, detect, or search objects in a perilous place. Similar applications involving coordination and cooperation can be seen in manufacture automation, wide-field monitoring, or even in data mining and machine learning. Many additional issues, which do not appear in the single-unit system, are raised in maneuvering a multi-agent system. The need to address the issues of, for instance, coordination [3], formation [4], flocking [5] and topology switching of these agents [6] manifests itself in the operations of a group of robots or drones. As control signals are continuously injected into the system, the state information of each unit is exchanged locally and then passed throughout the entire network. The performance of the system is thus determined by the effectiveness and efficiency of the state evolution. To ensure that any agent state can be effectively driven to any desired point in a finite time, the multi-agent system must be controllable. To efficiently drive these agent states, we expect the system to possess some internal structural property that leads to, possibly, relatively less energy consumption. The controllability problem is one of the main topics in the theory of multi-agent systems. The standard approach is to formulate the problem in the context of controlled linear and time-invariant (LTI) system following the consensus policy. This formulation relates the multi-agent system naturally to the graph model: the vertex set records the state variables and the edge set reflects the interactions between these variables, and results in the problem of Laplacian controllability of a graph. Due to Kalman, and to Popov, Belevitch and Hautus, the controllability can be checked from the rank fullness of the associated matrices [7, p. 145]. However, in the system whose number of agents reaches hundreds or more, one needs to check the rank of a very high-dimensional matrix and the computation result suffers from numerical inaccuracy. A natural approach is to leverage the graph model and applies the rich graph-theoretic results to the controllability analysis of MAS. Actually, a classical partition scheme in the graph theory was applied to catch the symmetry, in some sense, of the connecting structure and a sufficient condition based on the scheme for the system uncontrollability was proposed [8–10]. This graph-theoretic approach was later improved such that the lower and upper bounds based on the *distance partitions* and *almost equitable partitions*, respectively, can be derived for the controllable subspaces [11]. The elegance of the results is that the partitioning test is easily applicable to general graphs. However, this approach provides only partial results and leaves the controllability of many MASs inconclusive. Some researchers turned to explore the class of graphs that have specific connecting patterns, such as the paths [12], multichains [10,13], grids [14], circulant graphs [15] and complete graphs [16], and obtained abundant controllability results. The third line of research [17] is on the so-called zero forcing set/zero forcing number and their close relationship with the minimum rank problem of patterned matrices [18]. Though computing the zero forcing number and finding a minimum zero forcing set are in general not easy [19], zero forcing sets was used for controllability analysis of linear systems. A sufficient condition based on the sets for the controllability of graphs that are undirected and have specific property (e.g., the off-diagonal entries of

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the matrices carrying the structures of graphs have the same sign) was proposed [20]. This approach is powerful in determining the general controllability of a family of graphs, but, except for the extreme cases such as paths, circles and complete graphs, it is not widely applicable to other graphs with irregular distributions of vertex degrees when it comes to the Laplacian controllability. Though the Laplacian controllability of some classes of graphs is well-studied, they might demonstrate significantly different properties. For a k-vertex simple and connected graph, the vertex degree and the diameter range from 1 to $k-1$. These two parameters are intimately related to the construction cost and performance of the system, and are quite often subject to constraints in practice. For example, two classes of controllable graphs are the path graphs and a special kind of threshold graphs, known as the antiregular graphs [21], i.e., the class of simple and connected graphs having exactly two vertices with the same degrees. It was proved that any k-vertex graph in these two classes is controllable by a single controller. In Fig. 1 we compare the minimum-energy realizations of driving the states defined on these two graphs (both with 8 vertices and the same control vector) from $-10\,[\,1\,1\,1\,1\,1\,1\,1\,1\,]^T$ to $10\,[\,1\,1\,1\,1\,1\,1\,1\,1\,]^T$ in 10 seconds. The figure suggests that driving the states on an antiregular graph (shown in Fig. 2 (a)) is easier than on a path graph (in Fig. 2 (f)), in the sense that the consumed energy is less and the state transitions are smoother. Nevertheless, the ease comes at a price that the maximum vertex degree in an antiregular graph is up to $k-1$, compared to 2 in a path graph. The discussion above leads to a very practical problem: can we design a k-vertex graph that is controllable and at the same time its graph diameter or vertex degree is upper bounded?

In this paper we give a positive answer to the design problem above. Specifically, we propose

1. a simple method to combine a path graph and an antiregular graph such that the resulting graph is still single-input controllable and meets the constraint that imposes any meaningful upper bound on the graph diameter or on each vertex degree;
2. a simple method to identify the class of control vectors that guarantee the controllability of the combined graph. In particular, we show that this class is closely related to the one that renders the original antiregular graph controllable.

The contributions of our results are twofold. Firstly, we adopt a quite different approach to expanding the known class of controllable graphs from the conventional way that explores only the spectral properties of special Laplacian matrices [12,14–16,22,23]. Our approach shows the possibility to form a controllable graph from a hybrid of some well-known graphs. This possibility is crucial in exploring the controllability of general graphs whose Laplacian eigenspace properties are difficult to analyze. Secondly, our result serves as an example that the optimality, in some appropriate sense, and the controllability of a graph can be achieved simultaneously. This result is a pioneering example of a successful optimality–controllability co-design, and should motivate more studies on the network design issues for practical purposes.

The rest of this paper is organized as follows. In Section 2 we review the basic concepts in the graph theory and use them to model the controlled evolution of a multi-agent system in the context of a linear time invariant system. In Section 3 we present our main results on the method to construct a class of controllable graphs based on the combination of path graphs and antiregular graphs. Numerical examples are provided to illustrate the wide range of graph parameters, such as the diameters and maximum vertex degree, that the method can generate. The paper is concluded in Section 4 where some interesting future research topics are discussed.

2. Preliminaries

We begin with some standard notations and fundamental concepts used in the paper. Let $\mathbb{R}$ and $\mathbb{N}$ be the sets of real and natural numbers, respectively. $\mathbf{1}_k$ and $\mathbf{0}_k$ are the column vectors of 1's and 0's, respectively, with size k. Occasionally we skip the subscript when the context is clear. I is the identity matrix whose ith column vector is written as $\boldsymbol{e}_i$. The set of indices up to k is $\mathbb{I}_k := \{1, 2, \ldots, k\}$. The set difference of two sets S_1 and S_2 is $S_1 \setminus S_2$, defined as $\{s | s \in S_1, s \notin S_2\}$. $\lfloor x \rfloor$ and $\lceil x \rceil$ are the largest integer not greater than x and the smallest integer not less than x, respectively. Suppose P is a matrix of order k, meaning that $P \in \mathbb{R}^{k\times k}$, $\lambda \in \mathbb{R}$ and $\boldsymbol{v} \in \mathbb{R}^k$. $(\lambda, \boldsymbol{v})$ is called an eigenpair of P if $P\boldsymbol{v} = \lambda\boldsymbol{v}$. If $V = \mathbb{I}_k$ and E is a subset of $\{(v_1, v_2) \,|\, v_1, v_2 \in V\}$, then $\mathbb{G} := (V, E)$ describes a k-vertex graph where E is called the edge set of $\mathbb{G}$, and V the vertex set or node set. v_1 and v_2 are neighbors if $v_1, v_2 \in V$ and $(v_1, v_2) \in E$. The neighbor set $\mathcal{N}_v$ of the vertex v is $\{u \,|\, (v, u) \in E\}$. The degree of vertex v is defined as $|\mathcal{N}_v|$, i.e., the cardinality of $\mathcal{N}_v$ or the number of elements in $\mathcal{N}_v$. A vertex is called the terminal vertex if its degree is 1. It is called the dominating vertex if it is connected to all other vertices in the same graph. A path between vertices v_1 and v_2 is a subset $\{(v_1, u_1), (u_1, u_2), (u_2, u_3), \ldots, (u_{m-1}, u_m), (u_m, v_2)\}$ of E where $u_1, u_2, \ldots, u_m \in V$. The number of edges in the shortest path connecting v_1 and v_2 is called the distance between vertices v_1 and v_2. Among the distances between any pair of vertices of a graph, the greatest one is defined as the diameter of the graph. A graph is connected if for every two different vertices there exists a path connecting them. A graph is undirected if its edges have no orientation. Namely, if (v_1, v_2) is in the edge set of the graph then (v_1, v_2) is not an ordered pair. A graph is unweighted if its edges share the same weight. An undirected and unweighted graph is simple if it has no self-loops and no multiple edges. It is not difficult to see that a simple and connected graph should have at least two vertices with the same degree. If all vertices have the same degree, say d, then it is called a d-regular graph. In particular, it is called a complete graph if $d = k - 1$ where k is the number of all vertices in the graph. On the other hand, if there are exactly two vertices with the same degree, it is called an antiregular graph. Let d_i be the degree of the ith vertex of a k-vertex graph and satisfies $d_i \geq d_{i+1}$ for each $i \in \mathbb{I}_{k-1}$. Define $d_i^* := |j : d_j \geq i|$ and $\tilde{t} := |j : d_j \geq j|$. If

$$\sum_{i=1}^{k}(d_i + 1) = \sum_{i=1}^{k} d_i^*, \quad \forall k \in \{1, 2, \ldots, \tilde{t}\},$$

then the graph is called a threshold graph or maximal graph [24]. One can verify that a k-vertex antiregular graph must be a threshold graph and its only repeated vertex degree is $\lfloor \frac{k}{2} \rfloor$. A k-vertex simple and connected graph is called a path graph or simply a path if its diameter is up to $k-1$. It is easy to see that among the class of simple and connected graphs with the same number of vertices, if a graph has a dominating vertex, its diameter is at most 2 and its maximum vertex degree is the largest. A path graph has the largest diameter but its maximum vertex degree is the smallest. A complete graph has the smallest diameter but its maximum vertex degree is the largest. See [25,26] for more properties and concepts in the graph theory. Suppose a graph $\mathbb{G}(V, E)$ is given. Let $\mathcal{D}$ be the diagonal matrix with its ith diagonal term being the degree of the ith vertex. Let $\mathcal{A}$ be the adjacent matrix of the graph, meaning that its (i, j)th element is 1 if $(i, j) \in E$ and is 0 otherwise. The Laplacian matrix $\mathcal{L}$ of the graph is defined as

$$\mathcal{L} := \mathcal{D} - \mathcal{A}.$$

When we say the Laplacian eigenvalues or Laplacian eigenvectors of a graph, we mean the eigenvalues or eigenvectors of $\mathcal{L}$ corresponding to the graph. In our framework, we consider only the class of simple and connected graphs. Thus the corresponding