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Algebraic connectivity of network-of-networks having a graph product structure

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ABSTRACT

In this paper, we present a method for finding the algebraic connectivity of network-of-networks having a graph product structure. The network consists of several homogeneous (identical) subsystems connected with each other according to an interconnection graph. We show that the algebraic connectivity can be calculated from properties of graphs corresponding to the subsystem and the interconnection. Furthermore, we indicate that the algebraic connectivity of the entire system does not exceed those of the subsystem and the interconnection.

1. Introduction

Spectral graph theory [1] is a branch of algebraic graph theory, and it deals with the relation between properties of graph and Laplacian spectrum, i.e., the spectrum of graph Laplacian. In control theory, many systems can be modeled as graph, for instance, decentralized control systems, multi-agent systems [2,3], and so on. Thereby, various properties of the systems can be investigated through Laplacian spectrum of the corresponding graphs. For example, in consensus algorithms over networks, the algebraic connectivity of graph [4–6], that is, the second smallest eigenvalue of the graph Laplacian, characterizes how well the graph is connected and can be a measure for convergence rate.

In this paper, we propose a method for finding the algebraic connectivity of network-of-networks having a graph product structure. The networks are composed of several homogeneous subsystems, where their corresponding vertices are interconnected through an interconnection graph. We show that, without calculating Laplacian spectrum of graph corresponding to the entire system directly, the algebraic connectivity and the other eigenvalues can be obtained from properties of graphs corresponding to the subsystem and the interconnection. Also, we indicate that the algebraic connectivity of an entire system does not exceed those of the subsystem and the interconnection.

We remark that systems with network-of-networks structure have been investigated in several papers, e.g., [7,8], where controllability and observability have been considered. That is, the algebraic connectivity is not treated there. We also remark that we investigate the *exact value* of algebraic connectivity in this paper, while several *bounds* for the algebraic connectivity are known [6].

Our result is useful in designing network-of-networks with desired algebraic connectivity. The spectrum can be re-calculated easily when there is some change in the structure of the interconnection and/or the subsystems, or when some subsystems are added to or removed from the networks.

The outline of this paper is as follows. In Section 2, we address notation, mathematical basics, and graph Laplacian. Then we introduce network-of-networks to be considered and state our problem setting in Section 3. In Section 4, we present the main results of this paper. We show the case of a single interconnection , where there is only one connecting vertex on each subsystem, followed by the case of multiple interconnections where each subsystem has more than one connecting vertices. A few applications, where multi-agent systems represented by network-of-networks are shown in Section 5. An application to hierarchical graph product structure is also provided. In Section 6, we give some numerical examples which illustrate the main results. In Section 7, we make some concluding remarks.

Note that an earlier preliminary version of this paper is presented in [9], though it treats only single interconnection. It lacks a unified viewpoint to general network-of-networks and numerical examples.

2. Preliminaries

In this section, we address preliminaries such as notation, mathematical basics, and graph Laplacian.

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2.1. Notation and mathematical basics

The set of real numbers is denoted by $\mathbb{R}$. The cardinality of a set S is denoted by $|S|$. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, $\sigma(A) = \{\lambda_1(A), \ldots, \lambda_n(A)\}$ is its spectrum, where $\lambda_i(A)\,(i = 1, 2, \ldots, n)$ denote its eigenvalues in non-decreasing order, i.e., $\lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_n(A)$. Let A^T denote the transpose of A. The identity matrix of order n is denoted by $I_n \in \mathbb{R}^{n\times n}$. The kth column of I_n is denoted by $e(n, k)$. Let $A\otimes B$ denote the Kronecker product of matrices A and B. We recall some properties of Kronecker product. For $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$,

$$\det(A \otimes B) = (\det A)^m(\det B)^n \tag{1}$$

holds. When both AB and CD are defined,

$$(A \otimes C)(B \otimes D) = (AB) \otimes (CD) \tag{2}$$

holds.

2.2. Graph Laplacian

Consider an undirected graph $G = (V, E)$, where $V = \{1, \ldots, n\}$ and $E \subseteq V \times V$ denote the vertex set of G and the edge set of G, respectively. Its graph Laplacian $L(G) \in \mathbb{R}^{n\times n}$ is defined by

$$L(G) := D(G) - A(G),$$

where $A(G)$ is an adjacency matrix of G and $D(G)$ is a degree matrix of G. For $A(G) = [a_{ij}]$ with $i, j = 1, \ldots, n$, $a_{ij} = 1$ when $(i, j) \in E$ and $a_{ij} = 0$ otherwise. The matrix $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ with $d_i\,(i = 1, \ldots, n)$ denotes the degree of vertex i. For simplicity, with somewhat abuse of notation, instead of $\lambda_i(L(G))$ we use $\lambda_i(G)$ which denotes the ith eigenvalue of graph Laplacian $L(G)$.

3. Problem formulation

Let us first introduce graph theoretic modeling for network-of-networks having a graph product structure. We consider two undirected graphs $S = (V_S, E_S)$ and $C = (V_C, E_C)$, where $|V_S| = n_s$ and $|V_C| = n_c$. Here, V_S and V_C denote the vertex sets of S and C, while E_S and E_C denote the edge sets of S and C. Let a graph product $G = P(S, C, K)$ denote an undirected graph which is obtained by interconnecting corresponding vertices of every S with structure C. $K \subseteq V_S$ denotes the set of corresponding vertices of S which form interconnections. In other words, subsystems of G can be expressed by S while its structure of interconnection can be expressed by C and the set of connecting vertices can be expressed by K. The simplest examples are $G = C$ when S is a single vertex and $G = S$ when C is a single vertex. In Fig. 1 , we show an example of S, C, and G. In Fig. 1(a), connecting vertices are circled. We remark that the graph product defined above contains the rooted product [10] as well as the Cartesian product as a special case. In fact, the graph product becomes the rooted product if $|K| = 1$, while it becomes the Cartesian product if $|K| = n_s$.

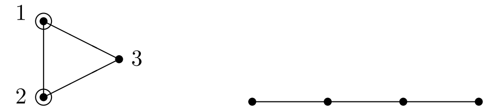
We consider network-of-networks whose structure can be modeled as the graph $G = P(S, C, K)$. The graph Laplacian of G can be written as

$$L(G) = I_{n_c} \otimes L(S) + L(C) \otimes M_{n_s}, \tag{3}$$

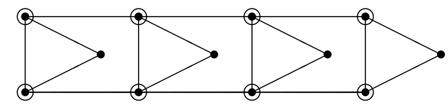
where $L(G) \in \mathbb{R}^{n\times n}$, $n = n_s n_c$, $M_{n_s} = \Sigma_{k\in K} M(n_s, k)$ with $M(n_s, k) = e(n_s, k)e(n_s, k)^T$.

In this paper, we investigate the following problem for network-of-networks.

Problem 1. For $G = P(S, C, K)$, find $\lambda_2(G)$ using properties of S and C, without calculating $\sigma(L(G))$ directly.



(a) Graph S. (b) Graph C.



(c) Graph $G = P(S, C, K)$.

Fig. 1. Examples of S, C, and $G = P(S, C, K)$ with $K = \{1, 2\}$.

4. Main results

4.1. The case of a single interconnection

In this subsection, we consider the case of a single interconnection, that is, the case of $|K| = 1$. Without loss of generality, we consider a vertex of S which forms an interconnection as vertex 1. In this case, we denote $G = P(S, C, \{1\})$ by $G = P(S, C)$ for simplicity.

We state main result of this subsection as the following theorem.

Theorem 4.1. *Suppose that S and C are undirected graphs. For the graph $G = P(S, C)$, all of the eigenvalues of $L(G)$ can be found by solving $\mu_i(p, L(G)) = 0$ which are given by*

$$\mu_i(p, L(G)) = \det[pI_{n_s} - \{L(S) + \lambda_i(C)M(n_s, 1)\}], \tag{4}$$

for $i = 1, \ldots, n_c$. Also, a specific eigenvalue can be found by

$$\lambda_{i+(k-1)n_c}(G) = \lambda_k(L(S) + \lambda_i(C)M(n_s, 1)), \tag{5}$$

where $i = 1, \ldots, n_c$ and $k = 1, \ldots, n_s$.

To prove Theorem 4.1, we need the following lemma, that is, Courant–Weyl inequalities (e.g. [11]).

Lemma 4.2. *Let X and Y be real symmetric matrices of order n. For $Z = X + Y$ and $0 \le i, j, i + j + 1 \le n$, inequalities*

$$\lambda_{n-i-j}(Z) \le \lambda_{n-i}(X) + \lambda_{n-j}(Y), \tag{6}$$

$$\lambda_{i+j+1}(Z) \ge \lambda_{i+1}(X) + \lambda_{j+1}(Y) \tag{7}$$

hold true.

Proof of Theorem 4.1. We first prepare an orthogonal matrix $U(C)$ which is constructed by unit-eigenvectors of $L(C)$. Then the graph Laplacian $L(C)$ can be diagonalized as

$$\Lambda(C) = U^T(C)L(C)U(C),$$

where $\Lambda(C) = \mathrm{diag}(\lambda_1(C), \ldots, \lambda_{n_c}(C))$. Now we consider the characteristic equation of $L(G)$ which is given by

$$\mu(p, L(G)) = \det(pI_n - L(G)). \tag{8}$$

Multiplying (8) by $\det(U^T(C)\otimes I_{n_s})$ and $\det(U(C)\otimes I_{n_s})$, we have

$$\begin{aligned}\mu(p, L(G)) &= \det(U^T(C) \otimes I_{ns}) \cdot \det(pI_n - L(G)) \det(U(C) \otimes I_{n_s}) \\ &= \det(pI_n - (L_1 + L_2)),\end{aligned} \tag{9}$$

where $L_1 = (U^T(C) \otimes I_{n_s})(I_{n_c} \otimes L(S))(U(C) \otimes I_{n_s})$ and $L_2 = (U^T(C) \otimes I_{n_s})(L(C) \otimes M(n_s, 1))(U(C) \otimes I_{n_s})$. The left-hand side of (9) does not change by multiplication because $\det(U^T(C) \otimes I_{n_s}) =$