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Partial exact controllability for wave equations

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ABSTRACT

We analyze the partial exact controllability problem for wave equations. The goal is to drive part of the solution to a destination at a given time. The main results are proved by employing the classical theory of propagation of singularities of wave equations and the compact uniqueness argument.

1. Introduction

Let (M, g) be a n-dimensional compact smooth Riemannian manifold with boundary ∂M. Let $T > 0$. Denote by Δ the Laplace–Beltrami operator on M.

The main purpose of this paper is to study the partial exact controllability problems of following wave equations

$$\begin{cases} y_{tt} - \Delta y = \chi_\omega f & \text{in } (0,T)\times M, \\ y = 0 & \text{on } (0,T)\times \partial M, \\ y(0) = y_0,\ y_t(0) = y_1 & \text{in } M \end{cases} \tag{1.1}$$

and

$$\begin{cases} z_{tt} - \Delta z = 0 & \text{in } (0,T)\times M, \\ z = \chi_\Gamma h & \text{on } (0,T)\times \partial M, \\ z(0) = z_0,\ z_t(0) = z_1 & \text{in } M. \end{cases} \tag{1.2}$$

In (1.1) (*resp.* (1.2)), $(y_0, y_1) \in H_0^1(M) \times L^2(M)$ (*resp.* $(z_0, z_1) \in L^2(M)\times H^{-1}(M)$), ω is an open subset of M (*resp.* Γ is an open subset of ∂M), $f \in L^2((0,T)\times\omega)$ (*resp.* $h \in L^2((0,T)\times\Gamma)$).

Let $M_0 \subset M$ be an open subset. The partial exact controllability of (1.1) and (1.2) with respect to M_0 is formulated as follows.

Definition 1.1. System (1.1) is said to be partially exactly controllable if for any given initial data $(y_0, y_1) \in H_0^1(M) \times L^2(M)$ and $(y_2, y_3) \in H_0^1(M) \times L^2(M)$, one can find a control $f \in L^2((0,T)\times\omega)$ such that the corresponding solution to (1.1) satisfies that $\big(\chi_{M_0} y(T), \chi_{M_0} y_t(T)\big) = \big(\chi_{M_0} y_2, \chi_{M_0} y_3\big)$.

Definition 1.2. System (1.2) is said to be partially exactly controllable if for any given initial data $(z_0, z_1) \in L^2(M) \times H^{-1}(M)$ and $(z_2, z_3) \in L^2(M) \times H^{-1}(M)$, one can find a control $h \in L^2((0,T)\times\Gamma)$ such that the corresponding solution to (1.2) satisfies that $\big(z(T)|_{M_0}, z_t(T)|_{M_0}\big) = \big(z_2|_{M_0}, z_3|_{M_0}\big)$.

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Remark 1.1. Here the notation $z_t(T)|_{M_0}$ means the restriction of the distribution $z_t(T)$ on M_0. Recall that one says an element $g \in H^{-1}(M)$ equals zero in M_0 if for any $\alpha \in H_0^1(M)$ which is supported in M_0, $\langle g, \alpha\rangle_{H^{-1}(M),H_0^1(M)} = 0$.

To guarantee the partial exact controllability of the system (1.1) and (1.2), we introduce the following two conditions, respectively.

Condition 1.1. *There is an $\omega_0 \subset\subset \omega$ such that every optics associated with the symbol of the wave operator issued at $t = 0$ and $x \in M_0$ intersects the set $(0,T)\times\omega_0$.*

Condition 1.2. *Every optics associated with the symbol of the wave operator issued at $t = 0$ and $x \in M_0$ intersects the set $(0,T)\times\Gamma$ at a non-diffractive point.*

We have the following results for the exact controllability of (1.1) and (1.2).

Theorem 1.1. *The system (1.1) is partially exactly controllable with respect to M_0, provided that Condition 1.1 holds.*

Theorem 1.2. *The system (1.2) is partially exactly controllable with respect to M_0, provided that Condition 1.2 holds.*

Following the idea of the Hilbert Uniqueness Method (see [1] for example), we introduce adjoint systems corresponding to (1.1) and (1.2), respectively. The adjoint system of (1.1) is

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } (0,T)\times M, \\ v = 0 & \text{on } (0,T)\times \partial M, \\ v(T) = v_0,\ v_t(T) = v_1 & \text{in } M. \end{cases} \tag{1.3}$$

Here

$$(v_0, v_1) \in H_{M_0} \triangleq \{(f_0, f_1) \in L^2(M)\times H^{-1}(M) : \\ \operatorname{supp}(f_0) \subset M_0\ \ \operatorname{supp}(f_1) \subset M_0\}.$$

The partial exact controllability of the system (1.1) with respect to M_0 is implied by the following observability estimate

$$|v_0|^2_{L^2(M)} + |v_1|^2_{H^{-1}(M)} \le C \int_0^T \int_\omega |v|^2 dxdt. \tag{1.4}$$

Here C is a constant which is independent of $(v_0, v_1) \in H_{M_0}$.

The adjoint system of (1.2) is

$$\begin{cases} w_{tt} - \Delta w = 0 & \text{in } (0,T) \times M, \\ w = 0 & \text{on } (0,T) \times \partial M, \\ w(0) = w_0,\ w_t(0) = w_1 & \text{in } M. \end{cases} \tag{1.5}$$

Here

$$(w_0, w_1) \in \widetilde{H}_{M_0} \overset{\triangle}{=} \{(g_0, g_1) \in H_0^1(M) \times L^2(M) : \text{supp}\,(g_0) \subset M_0,\ \text{supp}\,(g_1) \subset M_0\}.$$

The partial exact controllability of the system (1.2) with respect to M_0 is implied by the following observability estimate

$$|w_0|^2_{H_0^1(M)} + |w_1|^2_{L^2(M)} \le C \int_0^T \int_\Gamma \left|\frac{\partial w}{\partial \nu}\right|^2 d\Gamma dt, \tag{1.6}$$

where C is a constant which is independent of $(w_0, w_1) \in \widetilde{H}_{M_0}$.

We can show the following results.

Proposition 1.1. *The system (1.1) is partially exactly controllable with respect to M_0, provided that the inequality (1.4) holds.*

Proposition 1.2. *The system (1.2) is partially exactly controllable with respect to M_0, provided that the inequality (1.6) holds.*

Although proofs of Propositions 1.1 and 1.2 are standard dual argument, for the sake of completeness, we present them in Section 2.

From Propositions 1.1 and 1.2, in order to establish the partial exact controllability, we only need to prove the inequality (1.4) and (1.6). We have the following two results.

Theorem 1.3. *Inequality (1.4) is true, provided that Condition 1.1 holds.*

Theorem 1.4. *Inequality (1.6) is true, provided that Condition 1.2 holds.*

Controllability problems for wave equations have been studied extensively in the literature (see [1–9] and the rich references therein). Results in these papers are focused on the exact controllability problems for wave equations with some geometric conditions on the control domain. Particularly, by the Gaussian beam solution to wave equations (see [10]), we know that the geometric control condition is a necessary condition for the exact controllability of wave equations. In this paper, we handle the case when the geometric control condition does not hold. In this case, one cannot expect to get exact controllability. On the other hand, we show that part of the solution can be exactly controlled. Although the main idea of proofs of Theorems 1.3–1.4 are the same as the one in [3,4], we believe that it deserves to provide complete proofs for them.

2. Some preliminaries

In this section, we recall some useful results for the propagation of the singularities of the solution to a wave equation involved in a domain with nonempty boundary.

To begin with, we introduce some notations. For a smooth manifold N, we write $\dot{T}^*N$ for the set $T^*N \setminus (N \times \{0\})$, where T^*N denotes the cotangent bundle of N. Let us denote by Q the interior of the cylinder $(-\infty, +\infty) \times M$, by ∂Q the set $(-\infty, +\infty) \times \partial M$, and by $\overline{Q}$ the closure of Q. Let O be a neighborhood of Q such that $Q \subset\subset O$. Write $\dot{T}^*\overline{Q}$ for the restriction of $\dot{T}^*O$ on $\overline{Q}$, $\dot{T}_b^*Q$ for $\dot{T}^*Q \cup \dot{T}^*\partial Q$, and $\dot{T}^*_{\partial Q}$ for the conormal bundle to ∂Q in O. Let π be the canonical projection

$$\pi : \dot{T}^*\overline{Q} \setminus \dot{T}^*_{\partial Q} \to \dot{T}_b^*Q. \tag{2.1}$$

We equip $\dot{T}_b^*Q$ with the topology induced by π. For any $\xi \in T^*M$, denote by $|\xi|_g$ the norm of ξ with respect to the metric g. Let

$$Char(p) = \left\{(t, x, \tau, \xi) : (t, x, \tau, \xi) \in \dot{T}^*\overline{Q},\ \tau^2 - |\xi|_g^2 = 0\right\}$$

and

$$\Sigma_b = \pi(Char(p)).$$

The cotangent bundle to the boundary is the disjoint union of the elliptic set $\mathcal{E}$, the hyperbolic set $\mathcal{H}$ and the glancing set $\mathcal{G}$, which are consisted by points $\rho \in \dot{T}^*\partial Q$ such that p has, respectively, no zeros in $\pi^{-1}(\rho)$, two simple zeros in $\pi^{-1}(\rho)$ and double zeros in $\pi^{-1}(\rho)$.

Let $\rho_0 \in \mathcal{G}$ and $\beta_0 \in Char(p)$ such that $\pi(\beta_0) = \rho_0$. Let $\gamma : s \to \dot{T}_b^*Q$ be the integral curve of

$$H_p \overset{\triangle}{=} \Big(\frac{\partial p}{\partial \tau}\frac{\partial}{\partial t}, \frac{\partial p}{\partial \xi^1}\frac{\partial}{\partial x^1}, \dots, \frac{\partial p}{\partial \xi^n}\frac{\partial}{\partial x^n}, -\frac{\partial p}{\partial t}\frac{\partial}{\partial \tau}, -\frac{\partial p}{\partial x^1}\frac{\partial}{\partial \xi^1}, \dots, -\frac{\partial p}{\partial x^n}\frac{\partial}{\partial \xi^n}\Big)$$

such that $\gamma(0) = \beta_0$. Then, γ is tangent to ∂Q at β_0. Denote by $\mathcal{G}^k (k \ge 2)$ the set that the order of the contact of γ with ∂Q is exactly k. Let $\Sigma_b^{2,-}$ be the set such that $\beta(s) \in \dot{T}^*Q$ for $0 < |s| \le \delta$ with δ small enough, and $\Sigma_b^{2,+}$ the set such that $\beta(s) \notin \dot{T}^*Q$ for $0 < |s| \le \delta$, where $\delta > 0$ is arbitrary positive number.

Now recall the definition of a ray associated with the symbol of a wave operator.

Definition 2.1. Let $p = \tau^2 - |\xi|_g^2$. A ray associated with p is a continuous curve $\gamma : I \to \Sigma_b$, where $I \subset \mathbb{R}$ is an open interval, such that the following conditions hold.

1. If $\gamma(s_0) \in \Sigma_b \cap \dot{T}^*Q$, then γ is differentiable at s_0 and $\gamma'(s_0) = H_p(\gamma(s_0))$.
2. If $\gamma(s_0) \in \left(\Sigma_b \cap \dot{T}^*Q\right) \cup \Sigma_b^{2,-}$, then there is a $\delta > 0$ such that $\gamma(s_0) \in \Sigma_b \cap \dot{T}^*Q$ for $0 < |s - s_0| < \delta$.
3. If $\gamma(s_0) \in \Sigma_b^{2,+}$, then there is a $\delta > 0$ such that $\gamma(s) \in \Sigma_b^{2,+}$ for $|s - s_0| < \delta$. Further, γ is differentiable at s_0 (as a curve in $\Sigma_b^{2,+}$) and $\gamma'(s_0) = H_q(\gamma(s_0))$, where $q(t, x; \tau, \xi) = |\xi|_b^2 - \tau^2$, where $|\xi|_b$ is the length of $\xi \in T^*\Gamma$ for the metric induced by (M, g) on the boundary Γ.
4. If $\gamma(s_0) \in \mathcal{G}^3$ and $\{\tilde{\gamma}^+(s), \tilde{\gamma}^-(s)\}$ are the (at most) two points in $Char(p)$ such that $\pi(\tilde{\gamma}^+(s)) = \pi(\tilde{\gamma}^-(s)) = \gamma(s)$ and $\tilde{\gamma}^+(s_0) = \tilde{\gamma}^-(s_0)$, then
$$\lim_{s \to s_0} \frac{\tilde{\gamma}^+(s) - \tilde{\gamma}^+(s_0)}{s - s_0} = H_p(\tilde{\gamma}^+(s_0)),$$
$$\lim_{s \to s_0} \frac{\tilde{\gamma}^-(s) - \tilde{\gamma}^-(s_0)}{s - s_0} = H_p(\tilde{\gamma}^-(s_0)).$$

We call the projection of a ray γ to Q an optics associated with the symbol of the wave operator.

Let u be an extendible distribution on Q. Denote by $\text{WF}^s(u)$ the wavefront set of u and $\text{WF}_b^s(u)$ the wavefront set of u up to the boundary.

We recall the definition for non-diffractive point.

Definition 2.2. A point $\rho \in \dot{T}^*\partial Q$ is called non-diffractive if $\rho \in \mathcal{E} \cup \mathcal{H}$, or if $\rho \in \mathcal{G}$ and if $\beta \in Char(p)$ is the unique point such that $\pi(\beta) = \rho$, the ray γ through β with $\gamma(0) = \beta$ satisfies that for any $\varepsilon > 0$, there exists an $s \in (-\varepsilon, \varepsilon)$ such that $\gamma(s) \notin \dot{T}^*\overline{Q}$.