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On accessibility conditions for state space nonlinear control systems on homogeneous time scales

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ABSTRACT

An algebraic formalism for nonlinear control systems defined on homogeneous time scales is introduced. This formalism is based on the forward, backward shifts, delta and nabla operators of differential one-forms and vector fields. The accessibility conditions for considered control systems both in terms of vector fields and one-forms are presented.

1. Introduction

The fairly new area of time scale analysis combines the traditional areas of continuous- and discrete-time analysis into a single theory. The study of control systems on time scales brings together the study of traditional continuous- and discrete-time (or uniformly sampled) systems. This theory can also incorporate much less studied non-uniformly sampled systems by assuming non-homogeneous but regular time scale. The area has potential applications in engineering, ecology, medicine, economics, etc. The goal of this paper is to study accessibility property for nonlinear systems, defined on homogeneous time-scale. Note that accessibility, a special notion of controllability, is a fundamental system property that can be algebraically characterized using the concept of autonomous variable [1]. This property is closely related to reachability property: a continuous-time system is said to be accessible if its reachable set from almost all initial points has a non-empty interior [2].

Note that in [3] necessary and sufficient accessibility conditions were given for the set of nonlinear higher order input–output delta-differential equations, defined on a homogeneous time scale. In [3] the polynomial approach has been used to develop the accessibility condition that mimics the one from the linear theory, formulated in terms of the greatest common left divisor of two polynomial matrices. This paper focuses on systems, described by the state equations and presents two equivalent conditions. One of them is given in terms of differential one-forms and the other is related to the dimension of the distribution that annihilates the vector space of differential one-forms. In the continuous-time case they will lead to the vector space of one-forms with infinite relative degrees and strong accessibility distribution, respectively, see for instance [4]. The first condition is, in most cases, easier to apply. It allows to check generic accessibility property and also to find the autonomous variables of the system but is of no help in finding the singular points from which the system is not reachable. This is the main motivation for developing the second condition that has the advantage to determine the singular points from which the system is not reachable. Of course, this condition also allows to check generic accessibility and to find autonomous variables but is computationally more demanding. In the development of the second condition we use the recent definition of the nabla derivative of the vector field [5]. The algebraic approach used here assumes infinite dimensional spaces of vector fields, and as such has analogy with the Lie–Bäcklund approach in the continuous-time case [6].

Like in [3], our accessibility definition is based on the concept of autonomous variable, introduced by Pommaret (see for instance [1]) and found much use in nonlinear control literature [4,7–9]. The benefit (usefulness) of this concept depends on the

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fact that the definition is not related to specific system description nor specific subclasses of systems but is universally applicable in various situations.

The paper is organized as follows. Section 2 recalls the notion of time scale and its main concepts delta- and nabla-derivatives, provides the definition of homogeneous time scale and some notations used later on in the subsequent sections. Additionally, Section 2.2 presents the differential fields of meromorphic functions of system variables, associated with the considered control systems. Moreover, this section introduces the vector space of differential one-forms and the operators defined on this space are given. In Section 3 the properties of the subspaces of the vector space of differential one-forms and distributions of the dual vector space are presented. We show that the considered subspaces can be treated as codistributions on some sets. Moreover, the relation between the presented codistributions and distributions is given in Proposition 7 and Corollary 8. The main result, that is the necessary and sufficient accessibility conditions for a nonlinear control system, defined on a homogeneous time scale, are formulated and proved in Section 4. Finally, the example that illustrates our results is given.

2. Preliminaries

For the introduction to the calculus on time scales, see [10,11]. Here, we recall only those notions and facts that will be used later.

2.1. Time scale calculus

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. For $t \in \mathbb{T}$ the *forward* and *backward jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T} | s > t\}$, and $\rho(t) = \sup\{s \in \mathbb{T} | s < t\}$, respectively. In addition, we set $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ if there exists a finite $\max \mathbb{T}$, and $\rho(\min \mathbb{T}) = \min \mathbb{T}$ if there exists a finite $\min \mathbb{T}$. Then the *graininess* functions $\mu, \nu : \mathbb{T} \to [0, \infty)$ are defined by $\mu(t) = \sigma(t) - t$ and $\nu(t) = t - \rho(t)$, respectively. for all $t \in \mathbb{T}$. A time scale is called *homogeneous* if μ and ν are constant functions.

From now, we assume that $\mathbb{T}$ is a homogeneous time scale. The definitions and the properties of the *delta* and *nabla derivatives* f^{Δ} and f^{∇} of a real function f can be found, for instance, in [10,11] and they are also recalled in [5,12].

Remark 1. The delta and nabla derivatives are a natural extension of time derivative in the continuous-time case and respectively, forward and backward difference operators in the discrete-time case. Therefore, for $\mathbb{T} = \mathbb{R}, f^{\Delta}(t) = f^{\nabla}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s} = f'(t)$ and for $\mathbb{T} = \mathbb{Z}, f^{\Delta}(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)} = f(t+1) - f(t) =: \triangle f(t)$, where $\triangle$ is the usual forward difference operator, and $f^{\nabla}(t) = \frac{f(t)-f(\rho(t))}{\mu(t)} = f(t) - f(t-1) =: \nabla f(t)$, where ∇ is the usual backward difference operator.

Let $f^{[0]} := f$ and $f^{[1]} = f^{\Delta}$. For a function $f : \mathbb{T} \to \mathbb{R}$ we define higher order delta derivatives by $f^{[2]} : \mathbb{T} \to \mathbb{R}, f^{[2]} := \left(f^{\Delta}\right)^{\Delta}$ and $f^{[n]} : \mathbb{T} \to \mathbb{R}, f^{[n]} := \left(f^{[n-1]}\right)^{\Delta}$. $n \geqslant 3$. Note that for a homogeneous time scale $f^{[n]}$, $n \geqslant 1$ are uniquely defined for all $t \in \mathbb{T}$.

2.2. Differential fields

Now, we recall some definitions and facts given in [13,14] that will be used in the paper. The facts concerning the concepts of the σ_f-differential field can be found in [14] and of the ρ_f-differential field in [5]. Since $\mathbb{T}$ is homogeneous, the graininess functions are constant, i.e. $\nu = \mu = \text{const}$. Consider now the control system, defined on the homogeneous time scale $\mathbb{T}$,

$$x^{\Delta}(t) = f(x(t), u(t)), \tag{1}$$

where $(x(t), u(t)) \in \mathcal{X}\times\mathcal{U}$, $\mathcal{X}\times\mathcal{U}$ is an open subset of $\mathbb{R}^n\times\mathbb{R}^m$, $m \leq n$, x is a state, u is a control (input) of the system, and function $f : \mathcal{X}\times\mathcal{U} \to \mathbb{R}^n$ is analytic. Let us define $\widetilde{f}(x, u) := x + \mu f(x, u)$.

Assumption 1. Assume that there exists a map $\varphi : \mathcal{X}\times\mathcal{U} \to \mathbb{R}^m$ such that $\Phi = (\widetilde{f}, \varphi)^T$ is an analytic diffeomorphism[1] from the set $\mathcal{X}\times\mathcal{U}$ onto $\mathcal{X}\times\mathcal{U}$.

Assumption 1 means that from $(\bar{x}, z) = \left(\widetilde{f}(x, u), \varphi(x, u)\right) = \Phi(x, u)$ we can uniquely compute (x, u) as an analytic function of $(\bar{x}, z)$. For $\mu = 0$ this condition is always satisfied with $\varphi(x, u) = u$. In the case $\mu > 0$ the system (1) can be rewritten in the following equivalent form

$$x^{\sigma}(t) = \widetilde{f}(x(t), u(t)). \tag{2}$$

Then $z = \varphi(x, u) \in \mathbb{R}^m$ such that rank $\frac{\partial(\tilde{f},\varphi)}{\partial(x,u)} = n + m$.

For notational convenience, $(x_1, \ldots, x_n)$ will simply be written as x, and for $k \geq 0$ $\left(u_1^{[k]}, \ldots, u_m^{[k]}\right)$ will be written as $u^{[k]}$. For $i \leq k$, let $u^{[i..k]} := \left(u^{[i]}, \ldots, u^{[k]}\right)$. We assume that the control (input) applied to system (1) is infinitely many times delta differentiable, i.e. $u^{[0..k]}$ exists for all $k \geq 0$. Consider the infinite set of real (independent) variables $\mathcal{C} = \{x_i, i = 1, \ldots, n, u_j^{[k]}, j = 1, \ldots, m, k \geq 0\}$. Let $\mathcal{K}$ be the (commutative) field of meromorphic functions in a finite number of the variables from the set $\mathcal{C}$. Let $\sigma_f : \mathcal{K} \to \mathcal{K}$ be an operator defined by $\sigma_f(F)\left(x, u^{[0..k+1]}\right) := F\left(x + \mu f(x, u), u^{[0..k]} + \mu u^{[1..k+1]}\right)$, where $F \in \mathcal{K}$ depends on x and $u^{[0..k]}$. We assume that $(x, u) \in \mathcal{X}\times\mathcal{U}$ and the other variables are restricted in such a way that σ_f is well defined. Under Assumption 1, σ_f is injective endomorphism.

Additionally, the field $\mathcal{K}$ can be equipped with a delta derivative operator $\Delta_f : \mathcal{K} \to \mathcal{K}$ defined by

$$\Delta_f(F)\left(x, u^{[0..k+1]}\right) = \begin{cases} \dfrac{1}{\mu}\left[F\left(x + \mu f(x, u), u^{[0..k]} + \mu u^{[1..k+1]}\right) - F\left(x, u^{[0..k]}\right)\right], \\ \quad \text{if } \mu \neq 0, \\ \dfrac{\partial F}{\partial x}\left(x, u^{[0..k]}\right) f(x, u) + \displaystyle\sum_{k\geq 0} \frac{\partial F}{\partial u^{[0..k]}}\left(x, u^{[0..k]}\right) u^{[1..k+1]}, \\ \quad \text{if } \mu = 0, \end{cases} \tag{3}$$

where $F \in \mathcal{K}$ depends on x and $u^{[0..k]}$.

The more compact notations F^{σ_f} and F^{Δ_f} will be sometimes used instead of $\sigma_f(F)$ and $\Delta_f(F)$. Under Assumption 1, $\mathcal{K}$ endowed with the delta derivative Δ_f is a σ_f-differential field. For $\mu = 0$, $\sigma_f = \sigma_f^{-1} = \text{id}$ and $\mathcal{K}$ is inversive. It is always possible to embed $\mathcal{K}$ into an inversive σ_f-differential overfield $\mathcal{K}^*$, called the *inversive closure* of $\mathcal{K}$ [15] and extend σ_f to $\mathcal{K}^*$ so that $\sigma_f : \mathcal{K}^* \to \mathcal{K}^*$ is an automorphism. Let $\rho_f : \mathcal{K}^* \to \mathcal{K}^*$ be an operator defined by $\rho_f := \sigma_f^{-1}$. For $\mu \neq 0$, $\mathcal{K}^*$ is the field of meromorphic functions in variables $\mathcal{C}^* = \mathcal{C} \cup \{z_s^{\langle -\ell\rangle}, s = 1, \ldots, m, \ell \geq 1\}$ [13], where $z_i^{\langle -\ell\rangle} = \sigma_f\left(z_i^{\langle -\ell-1\rangle}\right)$ and $z_i = \varphi_i(x, u) = \sigma_f\left(z_i^{\langle -1\rangle}\right)$.

Let $z := (z_1, \ldots, z_m)$. Then $\left(\rho_f(x), \rho_f(u)\right) = \Psi\left(x, z^{\langle -1\rangle}\right)$, where Ψ is a certain vector valued function, determined by $\Phi(x, u)$. The extension of operator Δ_f to $\mathcal{K}^*$ can be made in analogy to (3). Such operator Δ_f is now a σ_f-derivation of $\mathcal{K}^*$ since it satisfies the following generalized Leibniz rule $\Delta_f(FG) = \Delta_f(F)G + \sigma_f(F)\Delta_f(G)$. Additionally, the field $\mathcal{K}^*$ can be equipped with a nabla derivative operator $\nabla_f : \mathcal{K}^* \to \mathcal{K}^*$ defined by

$$\nabla_f(F) := \begin{cases} \dfrac{1}{\mu}\left[F - \rho_f(F)\right], & \text{if } \mu \neq 0 \\ \Delta_f(F), & \text{if } \mu = 0. \end{cases} \tag{4}$$

[1] This assumption guarantees that the system $x^{\sigma} = \tilde{f}(x, u)$ is submersive, that is generically rank $\frac{\partial \tilde{f}(x,u)}{\partial(x,u)} = n$.