Commun Nonlinear Sci Numer Simulat 57 (2018) 125–135

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Finite-dimensional Liouville integrable Hamiltonian systems generated from Lax pairs of a bi-Hamiltonian soliton hierarchy by symmetry constraints

Solomon Manukure

Department of Mathematics, The University of Texas at Austin, Austin, TX 78712-1202, USA

A R T I C L E I N F O

Article history:
Received 18 July 2017
Accepted 13 September 2017
Available online 20 September 2017

MSC:
37K05
37K10
35Q53

Keywords:
Finite-dimensional system
Liouville integrability
Bi-Hamiltonian system
Involutive solutions
Soliton hierarchy

A B S T R A C T

We construct finite-dimensional Hamiltonian systems by means of symmetry constraints from the Lax pairs and adjoint Lax pairs of a bi-Hamiltonian hierarchy of soliton equations associated with the 3-dimensional special linear Lie algebra, and discuss the Liouville integrability of these systems based on the existence of sufficiently many integrals of motion.

1. Introduction

The Hamiltonian formalism plays a very important role in both pure and applied mathematics. For example, in the field of mathematical physics, Hamiltonian systems can be used to model dynamical systems that evolve without dissipation. In the last couple of decades there has been a considerable amount of interest in the study of Hamiltonian systems, especially systems of nonlinear evolution equations that can be presented in bi-Hamiltonian forms. These systems have an important attribute of allowing for an interplay between symmetries and conservation laws which forms a significant component of the concept of integrability of differential equations in the sense of Liouville. Another important attribute is that finite-dimensional Liouville integrable Hamiltonian systems can be produced when a hierarchy of bi-Hamiltonian soliton equations is restricted to a finite-dimensional manifold invariant with respect to all equations in the hierarchy [1]. In this article, we explore this property in relation to a bi-Hamiltonian soliton hierarchy associated with the three-dimensional special linear Lie algebra by using the so-called *Bargmann symmetry constraint* [2–4]. The Bargmann symmetry constraint method involves a nonlinearization procedure in which the Lax pairs and adjoint Lax pairs of a soliton hierarchy are nonlinearized into a hierarchy of finite-dimensional Liouville integrable Hamiltonian systems. This technique has been studied for many years and has been successfully applied to many soliton hierarchies to obtain finite-dimensional Liouville Hamiltonian systems.

E-mail addresses: smanukure@math.utexas.edu, smanukure@mail.usf.edu

https://doi.org/10.1016/j.cnsns.2017.09.016

The paper is organized as follows. In the next section, we introduce a 2×2 matrix spectral problem and consequently derive its hierarchy of soliton equations via the zero curvature formulation scheme [5,6]. As remarked earlier, we take the underlying Lie algebra to be the three-dimensional real Lie algebra $\mathrm{sl}(2,\mathbb{R})$ which consists of 2×2 traceless real matrices. We choose the matrices

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{1.1}$$

as its basis and

$$\widetilde{\mathrm{sl}}(2,\mathbb{R}) = \left\{ \sum_{i\geq 0}^{\infty} A_i \lambda^{n-i} | A_i \in \mathrm{sl}(2,\mathbb{R}), \ i \geq 0 \ , n \in \mathbb{Z} \right\}. \tag{1.2}$$

as its associated loop algebra. In Section 3, we carry out the binary nonlinearization technique to obtain two finite-dimensional Hamiltonian systems with respect to the spatial and temporal variables. We then show that these equations are Liouville integrable in the sense that they possess sufficiently many functionally independent and mutually involutive integrals of motion. Lastly, we present involutive solutions to the soliton hierarchy. In the final section, we give some concluding remarks.

2. Soliton hierarchy and its Hamiltonian formulation

To begin with, let us prove the following theorem.

Theorem 2.1. *The spectral problem*

$$\phi_x = U\phi = U(u,\lambda)\phi \in \widetilde{\mathrm{sl}}(2,\mathbb{R}), \tag{2.3}$$

with spectral matrix

$$U = \lambda q e_1 + (\lambda^2 + \lambda p)e_2 + (-\lambda^2 + \lambda p)e_3 = \begin{bmatrix} \lambda q & \lambda^2 + \lambda p \\ -\lambda^2 + \lambda p & -\lambda q \end{bmatrix}, \tag{2.4}$$

produces a hierarchy of soliton equations

$$u_{t_m} = K_m = \Phi^m \begin{bmatrix} p_x \\ q_x \end{bmatrix}, \ m \geq 0, \tag{2.5}$$

where the operator Φ *is given by*

$$\Phi = \begin{bmatrix} \partial p \partial^{-1} p & \frac{1}{2}\partial + \partial p \partial^{-1} q \\ -\frac{1}{2}\partial + \partial q \partial^{-1} p & \partial q \partial^{-1} q \end{bmatrix}, \ \partial = \frac{\partial}{\partial x}. \tag{2.6}$$

Proof. We employ the zero-curvature formulation scheme to prove the above theorem. First, we define W in the stationary zero-curvature equation

$$W_x = [U, W] \tag{2.7}$$

as

$$\begin{aligned} W &= ce_1 + (a+b)e_2 + (a-b)e_3 \\ &= \sum_{i\geq 0} \begin{bmatrix} c_i \lambda^{-2i-1} & a_i \lambda^{-2i-1} + b_i \lambda^{-2i} \\ a_i \lambda^{-2i-1} - b_i \lambda^{-2i} & -c_i \lambda^{-2i-1} \end{bmatrix} \in \widetilde{\mathrm{sl}}(2,\mathbb{R}). \end{aligned} \tag{2.8}$$

Eq. (2.7) then gives the relations

$$a_m = \frac{1}{2} c_{m-1,x} + p b_m, \ c_m = q b_m - \frac{1}{2} a_{m-1,x} \ \text{ and } \ b_{m,x} = p a_{m-1,x} + q c_{m-1,x}, \ m \geq 1, \tag{2.9}$$

with initial values

$$a_0 = p, \ b_0 = 1, \ c_0 = q. \tag{2.10}$$

Now, we introduce a sequence of Lax operators

$$V^{[m]} = \lambda(\lambda^{(2m+1)} W)_+, \ m \geq 0, \tag{2.11}$$

and as a result, the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \ m \geq 0, \tag{2.12}$$