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Research paper

Classical solution for the linear sigma model

Danilo V. Ruy

Departamento de Física, Universidade Federal da Paraíba, 58051-970 João Pessoa PB, Brazil

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A B S T R A C T

In this paper, the linear sigma model is studied using a method for finding analytical solutions based on Padé approximants. Using the solutions of two and three traveling waves in 1+3 dimensions we found, we are able to show a solution that is valid for an arbitrary number of bosons and traveling waves.

1. Introduction

Field theory is a well established way to describe interactions among particles, but usually it is very challenging to find analytical classical solutions in 1+3 dimensions. In this paper, we will study the bosonic interaction of the linear sigma model, which was proposed in [1] as a model to describe the pion-nucleon interaction. The model we are considering possess N_ϕ bosonic fields whose Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \mathbf{\Phi})^2 - \frac{1}{2}m^2(\mathbf{\Phi})^2 - \frac{\lambda}{4}((\mathbf{\Phi})^2)^2$$

where $(\mathbf{\Phi})^2 \equiv \mathbf{\Phi}\cdot\mathbf{\Phi}$ and $\mathbf{\Phi} = (\phi_1, \phi_2, \dots, \phi_{N_\phi})$. The Euler-Lagrange equation for this Lagrangian yields the system of equations

$$\phi_{i,tt} - \nabla\phi_i + m^2\phi_i + \lambda\left(\phi_i^3 + \phi_i \sum_{j=1}^{N_\phi - 1} \phi_{i+j}^2\right) = 0, \qquad i = 1, \dots, N_\phi, \tag{1}$$

where we use the notation $\phi_i = \phi_{i+N_\phi}$ and the metric $(+---)$. The purpose of this paper is to seek classical solutions for this system in the form of traveling waves. In order to achieve that, we will employ the algorithm based on PadÃ approximants presented in [19,20]. This method has the advantage of needing less undetermined constants than other methods in the literature [2–18], such that we can earn efficiency in the processing of data.

We briefly recall the algorithm from [20] in Section (2) and we apply it to the linear sigma model in Section (3). Using the solutions for two and three traveling waves we found, we are able to show a solution for an arbitrary number of

E-mail addresses: daniloruy@ift.unesp.br, danilo_ruy@hotmail.com

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traveling waves and bosons in Section (3.3). Section (4) is devoted to show a graphical illustration of the analytical solution we found. Finally, in Section 5, we compare the Padé approximant approach with the multiple exp-method in order to see the benefits of each method.

2. Review of the algorithm

Consider a system of N_e equation in D dimensions

$$E_k(x^\mu, \phi_i, \partial_\mu \phi_i, \ldots; \mathcal{S}_0) = 0, \qquad k = 1, \ldots, N_e. \tag{2}$$

where $\mathcal{S}_0$ is the set of all parameters of the model. Now suppose there is at least one solution of this system which can be written in terms of N_ρ functions we know the first derivative, i.e.

$$\begin{aligned} \phi_i(x^\mu) &= \hat{\phi}_i(\rho_1, \ldots, \rho_{N_\rho}), & i &= 1, \ldots, N_e \\ \partial_\mu \rho_k &= \mathcal{F}_{\mu,k}(\rho_1, \ldots, \rho_{N_\rho}; \mathcal{S}_1), & \mu &= 1, \ldots, D \qquad k = 1, \ldots, N_\rho \end{aligned}$$

where $\mathcal{F}_{\mu,k}(\rho_1, \ldots, \rho_{N_\rho}; \mathcal{S}_1)$ is determined by the functions we choose for ρ_k and $\mathcal{S}_1$ is the set of constants introduced by these functions. This ansatz transforms the system (2) as

$$E_k(x^\mu, \phi_i, \partial_\mu \phi_i, \ldots; \mathcal{S}_0) = \hat{E}_k(\rho_k, \hat{\phi}_i, \partial_k \hat{\phi}_i, \ldots; \mathcal{S}_0 \times \mathcal{S}_1) = 0, \qquad k = 1, \ldots, N_e. \tag{3}$$

Suppose there is a solution that is regular when $\rho_k \to 0$ such that we can calculate the multivariate Taylor expansion at $\rho = 0$, i. e.

$$\hat{\phi}_i(\rho) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_{N_\rho}=0}^{\infty} c^{(i)}_{j_1, j_2, \ldots, j_{N_\rho}} \prod_{d=1}^{N_\rho} \rho_d^{j_d}, \qquad i = 1, \ldots, N_e \tag{4}$$

In the expansion (4), we may also impose a set of constraints $\psi_i(\mathcal{S}_0 \times \mathcal{S}_1) = 0$ on the undetermined constants such that the series does not truncate in the vacuum case. We will call $\mathcal{S}_2$ the set of all undetermined $c^{(i)}_{j_1, j_2, \ldots, j_{N_\rho}}$ and $\mathcal{S} = \mathcal{S}_0 \times \mathcal{S}_1 \times \mathcal{S}_2$ the set of all constants that remains undetermined. Let us do the scaling transformation $\rho \to \xi \rho$ such that we can rearrange the Taylor expansion as

$$\hat{\phi}_i(\xi \rho) = \sum_{n=0}^{L_i + M_i} a_{i;n}(\rho) \xi^n + \mathcal{O}(\xi^{L_i + M_i + 1}), \qquad i = 1, \ldots, N_e,$$

$$a_{i;n}(\rho) = \sum_{j_1=0}^{n} \sum_{j_2=0}^{n-j_1} \cdots \sum_{j_{N_\rho -1}=0}^{n - \sum_{r=1}^{N_\rho - 2} j_r} c^{(i)}_{j_1, j_2, \ldots, j_{N_\rho - 1}, n - \sum_{r=1}^{N_\rho - 1} j_r} \left(\prod_{d=1}^{N_\rho - 1} \rho_d^{j_d} \right) \rho_{N_\rho}^{n - \sum_{r=1}^{N_\rho - 1} j_r}.$$

The coefficients of the expansion (4) need be calculated until we determine all $a_{i;\, n}(\rho)$ up to order $L_i + M_i$. Now, for a chosen L_i and M_i, we can calculate the Padé approximant for the field using ξ as variable, i.e.

$$\hat{\phi}_i(\xi \rho) = \frac{P^{(i)}_{\rho, L_i}(\xi; \mathcal{S})}{Q^{(i)}_{\rho, M_i}(\xi; \mathcal{S})} + \mathcal{O}(\xi^{L_i + M_i + 1}). \tag{5}$$

The Padé approximants consist in an approximation of a function as a ratio of two polynomials given by

$$[L_i/M_i]^{(i)}_\rho(\xi; \mathcal{S}) \equiv \frac{P^{(i)}_{\rho, L_i}(\xi; \mathcal{S})}{Q^{(i)}_{\rho, M_i}(\xi; \mathcal{S})} = \frac{\sum_{j=0}^{L_i} p^{(i)}_j(\rho; \mathcal{S}) \xi^j}{1 + \sum_{j=1}^{M_i} q^{(i)}_j(\rho; \mathcal{S}) \xi^j}$$

whose coefficient are calculated by the system of equations

$$\sum_{r+s=j} a_{i;r}(\rho) q^{(i)}_s(\rho; \mathcal{S}) - p^{(i)}_j(\rho; \mathcal{S}) = 0, \qquad j = 0, 1, \ldots, L_i + M_i.$$

Now we would like to find the subset $\hat{\mathcal{S}} \subset \mathcal{S}$ which makes the approximation (5) become an exact solution for the system (2) when $\xi = 1$, i.e.

$$\hat{\phi}_i(\rho) = \left. \frac{P^{(i)}_{\rho, L_i}(\xi; \hat{\mathcal{S}})}{Q^{(i)}_{\rho, M_i}(\xi; \hat{\mathcal{S}})} \right|_{\xi = 1}. \tag{6}$$

Substituting (6) in (3), we can rewrite the system as

$$\hat{E}_k(\rho_k, \hat{\phi}_i, \partial_k \hat{\phi}_i, \ldots; \mathcal{S}_0 \times \mathcal{S}_1) = \frac{\sum_{n=0}^{\Lambda} \hat{E}_{k;n}(\hat{\mathcal{S}})}{D_k(\hat{\mathcal{S}})} = 0,$$