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Research paper

# Seamless variation of isometric and anisometric dynamical integrity measures in basins's erosion

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ABSTRACT

Anisometric integrity measures defined as improvement and generalization of two existing measures (LIM, local integrity measure, and IF, integrity factor) of the extent and compactness of basins of attraction are introduced. Non-equidistant measures make it possible to account for inhomogeneous sensitivities of the state space variables to perturbations, thus permitting a more confident and targeted identification of the safe regions. All four measures are used for a global dynamics analysis of the twin-well Duffing oscillator, which is performed by considering a nearly continuous variation of a governing control parameter, thanks to the use of parallel computation allowing reasonable CPU time. This improves literature results based on finite (and commonly large) variations of the parameter, due to computational constraints. The seamless evolution of key integrity measures highlights the fine aspects of the erosion of the safe domain with respect to the increasing forcing amplitude.



## 1. Introduction

In nonlinear dynamics the possible coexistence of multiple attractors for a given system is well known and, indeed, it represents the common situation named multistability [1–3]. A generic attractor, be it a stationary (equilibrium) point, a limit-cycle, a quasi-periodic or a chaotic motion, can be detected by inspecting orbits of the system as the time goes to infinity. Its basin of attraction is the subset of the phase space, i.e. the set of initial conditions, that converges to it forward in time. Because of multistability, the phase space is the union of various basins of attraction, one per attractor, which do not intersect with each other, since the forward evolution in time is unique [4].

By modifying the system parameters the position of attractors change, and they can also bifurcate leading to different attractors. Related basins of attraction are altered as well: basins deform and reshape, new ones can appear or existing be destroyed. Even if system parameters slightly change, a rapid erosion or a stratification of the basin can occur [5,6], often as a consequence of homoclinic intersection of stable and unstable manifolds [4].

A basin can be a substantially large region of the phase space but, if its structure is highly intertwined or characterized by fractal boundaries, with a "small" compact part around the attractor, the long-term predictability depends on uncertainties

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in the initial conditions specification [7]. In this case, the stability of an attractor in classical Liapunov's sense, i.e. against infinitesimal disturbances, does not have a practical engineering validity as noisy environments and imperfections must be accounted for. The "practical stability" is needed instead of the "theoretical stability", and the safety of the system is related to the capability to accommodate "moderate" (small but not infinitesimal) perturbations without undesired consequences [8].

The loss of engineering robustness due to larger excitations has been analyzed in the fundamental work of Thompson [9]. The concept of dynamical integrity is therein introduced and applied to a mechanical oscillator where the escape from a cubic potential well beyond the hilltop saddle is presented. Several works have then examined the erosion of basins of attraction by focusing on the practical stability reduction and the consequent vulnerability to disturbances. The contact of a single-well attractor in the twin-well Duffing oscillator with the boundaries of its basin is examined in [10]. The control, shift and reduction of the erosion of the basins of attraction has been studied with the aim of the elimination of homoclinic tangencies and regularization of fractal basins both in macro [11] and micro systems [12].

The estimation of the safety of basins of attraction plays a key role in a system design. The issue has been addressed by Soliman and Thompson [13] and in the last years Rega and Lenci [14] investigated different measures able to quantify an admissible set of initial conditions of the phase space.

In dynamical integrity analyses it is crucial to study how the basins of attraction, and their integrity measures, evolve by varying a parameter of the system. This provides the so-called *integrity profiles* [15,16], that are very helpful in understanding the degradation of robustness, and allow to fix some thresholds for safety.

Drawing integrity profiles requires building many basins of attractions, and thus it is very time consuming. It is for this reason that integrity profiles are usually built only for "discrete" values of the varying parameter. While being sufficient for an overall understanding of the system behaviour, this misses some important aspects of the dynamics like, for example, sudden jumps (due to the appearance of a new attractor inside the basin of a previous attractor) and locally strange behaviours (commonly related to minor or rare attractors [17,18]). It is thus extremely useful to have seamless integrity profiles, which provide information otherwise lost.

This is the first goal of this paper, where "continuous" curves are obtained by building a huge number of basins of attraction. This has been possible, in reasonable time, by using high performance computation, and in particular parallel computing, adopting in the present analysis algorithms developed by the authors for elaborating basins of attractions of high dimensional systems [19,20].

In previous works perturbations of initial conditions in displacement and velocity are assumed to be equally important, and are then considered of the same level. In fact, previously proposed (isometric) integrity measures do not distinguish between displacement and velocity, or more generally between different coordinates in phase space. However, it may happen that a system is more susceptible, say, to velocity rather than to displacement, e.g. under the presence of an impulsive excitation. It is thus useful to introduce measures that take into account this aspect. As a byproduct, this allows us to better deal with mechanical degrees of freedom (displacements vs velocities) that have different physical dimensions, being thus affected by the choice of the system of measurement. This is the second goal of this paper. Anisometric integrity measures weighting in a different way each coordinate of the phase space are introduced and deeply investigated.

Both seamless integrity profiles and the performance of different integrity measures are investigated with reference to the global dynamics of a two-well bistable Duffing oscillator. The paper is organized as follows. Anisometric integrity measures are defined in Section 2 as generalization of the underlying isometric ones. The considered paradigmatic model is illustrated in Section 3. Seamless variation of all measures for competing resonant and non-resonant attractors of the two-well Duffing oscillator is comparatively addressed in Section 4, by dwelling on their features and capability to identify regions of the phase space with predictable outcomes. In particular, their evolution in a 3D phase-parameter space is addressed in Section 5. The last section draws some conclusions and hints for future developments.

## 2. Isometric and anisometric dynamical integrity measures

If not preliminarily defining/identifying an actual safe basin, as discussed in [8,14], the most straightforward integrity measure of a basin of attractor is represented by the Global Integrity Measure (*GIM*), namely by its hyper-volume, that reduces to an area in 2D problems. The *GIM* cannot be deemed a prudent indicator because it accounts for both compact and fractal, i.e. dynamically non-integer, parts of the basin. A more refined measure able to get rid of the unsafe fractal tongues from the integrity evaluation is the Integrity Factor (*IF*) [14]. The IF considers only the compact part of the basin and calculates the largest hyper-sphere within (circle in 2D cases). A more conservative measure is represented by the Local Integrity Measure (*LIM*), whose definition retraces the IF, being again the radius of the largest hyper-sphere entirely contained in the basin but constrained to be centred at the attractor of reference. A less used measure, expression of the safe attractor stability, is the impulsive integrity measure (*IIM*). It is the distance between the attractor and the nearest boundary of the basin along the direction related to the generalized velocity. $IIM^{\pm}$ indicates the minimum of the distances along positive ($IIM^{+}$) and negative ($IIM^{-}$) directions.

As it can be perceived, all the aforementioned measures do not account for inhomogeneous sensitivities of the state space variables but, in practical applications, a system could be primarily affected by perturbations in one of its characteristic quantities (e.g. velocity, position, etc.). The local integrity measure is thus generalized with the introduction of the *Anisometric Local Integrity Measure* (*ALIM*) that is non-equidistant in the state-space coordinates. It is defined as