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Enhanced group analysis of a semi linear generalization of a general bond-pricing equation

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A B S T R A C T

The enhanced group classification of a semi linear generalization of a general bond-pricing equation is carried out by harnessing the underlying equivalence and additional equivalence transformations. We employ that classification to unearth the particular cases with a larger Lie algebra than the general case and use them to find non trivial invariant solutions under the terminal and the barrier option condition.

1. Introduction

Recently, Sinkala et al. introduced a general bond–pricing equation

$$u_t + \frac{1}{2}\rho^2 x^{2\gamma} u_{xx} + (\alpha + \beta x - \lambda \rho x^{\gamma}) u_x - xu = 0, \tag{1}$$

where α, β, γ, ρ and λ are real constants and $u = u(x, t)$ with $x > 0$ (see [1–3] and references therein). The equation has an interesting property: it encompasses some of the classical models of financial mathematics, namely the Longstaff model ($\gamma = \delta = 1/2, \alpha = \rho^2/4$), the Vasicek model ($\gamma = \delta = 0, \beta \neq 0$) and the Cox–Ingersoll–Ross model ($\gamma = \delta = 1/2, \lambda = 0$) [4–6].

In the present work we study — from a mathematical viewpoint — a semi linear generalization of Eq. (1),[1] that is

$$u_t + \frac{1}{2}\rho^2 x^{2\gamma} u_{xx} + (\alpha + \beta x - \lambda \rho x^{\delta}) u_x - f(x, u) = 0, \quad \rho \neq 0, \quad \delta \neq 0, 1, \tag{2}$$

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[1] Immediately one can observe that Eq. (2) contains also the celebrated Black–Scholes–Merton model [7] when $\alpha = \lambda = 0, \gamma = 1$ and $f(x, u) = \beta u$.

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using the analytic machinery provided by symmetry analysis. Our aim is to find the complete group classification of Eq. (2) and use it to propose nonlinear models with potential interest to financial mathematics and other scientific areas that use models like (1). In other words, to find specific instances of the function $f(x, u)$ that enlarge the Lie point symmetry group $\mathcal{G}$ of the most general case: for an arbitrary function f [8, p. 178]. In order to achieve that goal we use elements from the enhanced group analysis, namely equivalence and additional transformations, in order to simplify the task at hand [9,10].

Apart from indicating potential models, a larger set of symmetries plays a prominent role when one also considers a set of initial or boundary conditions along the PDE. In that case the chance to find a subset of symmetries that admit the initial or boundary problem as a whole increases, and as a consequence, our capacity to construct invariant solutions for the initial or boundary problem.

Since we obtained Eq. (2) as a generalization of the model (1) we shall also adopt two specific problems associated with it,

1. The terminal condition

$$u(x, T) = 1, \tag{3}$$

and

2. The barrier option condition

$$u(H(\tau), \tau) = R(\tau), \tag{4a}$$

$$u(x, T) = \max(x - K, 0), \tag{4b}$$

where T is the terminal time and $\tau = t - T$ the time to expiry.

In the context of financial mathematics, and Eq. (1), the former describes the evolution of standard or "vanilla" products [7,11–13], while the latter an exotic type of products [14–16]. Moreover, a common assumption for the function H — termed also as the *barrier* function — is to have the exponential form

$$H(\tau) = bKe^{-a\tau}, \tag{5}$$

where $a \geq 0$, $0 \leq b \leq 1$ and $\tau = t - T$ [15, p. 187]. Besides, like in [16], by similarity solutions admitting the barrier condition we mean solutions that obey (4a) only. The reason for this is that we are interested to study the case where x, the stock price, hits the barrier function, and as consequence u, the option, is terminated. Hence, the terminal condition (4b) has no relevance.[2]

As we previously mentioned the key analytical tool used in this work is the symmetry analysis of the Eq. (2). One of the advantages of this approach is that it provides a well-defined algorithmic procedure which essentially enables one to find the involved linearizing transformations, conservation laws, invariant solutions, etc. In fact, various works on the classical financial models mentioned above have a connection with the heat equation [17,18]. As probably expected, Eq. (2) shares a connection with the heat equation. More precisely we prove here that Eq. (2) is linked with the heat equation with nonlinear source,

$$u_t = u_{xx} + \bar{f}(x, u). \tag{6}$$

Furthermore, we show that the exponential form (5) usually utilized in the literature is admitted by the symmetries found. A fact that reinforces the *physical* importance of this specific choice by the specialist of the field.

At this point we would like to mention that for all the calculations involved the symbolic package SYM for Mathematica was extensively used, both for the interactive manipulation of the found symmetries as well as for determining the equivalence transformation and the classification of Eq. (2) [19].

This paper is organized as follows. In Section 2 we obtain the set of the continuous equivalence transformations and with their help the complete group classification of Eq. (2). In Section 3 we give specific examples of invariant solutions under the specific boundary problems which we study: the "vanilla" option and the barrier option. Finally, in Section 4 we discuss the results of the present work.

2. Group classification

In this section we proceed with the group classification of Eq. (2) following the same principles as in our previous works, see [20,21]. First, the best representative for the class of equations (2) is obtained utilizing its equivalence algebra. To do that, we construct the continuous part of the equivalence group and with its help we zero out as many arbitrary elements as possible.

[2] If x do not hit the barrier the option is exercised at $\tau = 0$, so in fact we deal with a terminal condition.