Commun Nonlinear Sci Numer Simulat 54 (2018) 233–247

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Solving of the coefficient inverse problems for a nonlinear singularly perturbed reaction-diffusion-advection equation with the final time data[☆]

D.V. Lukyanenko[a,*], M.A. Shishlenin[b,c,d], V.T. Volkov[a]

[a] Department of Mathematics, Faculty of Physics, Lomonosov Moscow State University, Moscow 119991, Russia
[b] Sobolev Institute of Mathematics, Novosibirsk 630090, Russia
[c] Institute of Computational Mathematics and Mathematical Geophysics, Novosibirsk 630090, Russia
[d] Novosibirsk State University, Novosibirsk 630090, Russia

ARTICLE INFO

Article history:
Received 13 August 2016
Revised 24 December 2016
Accepted 3 June 2017
Available online 6 June 2017

MSC:
65M32
65L04
65L12
65L20
65M20
35G31

Keywords:
Singularly perturbed problem
Interior and boundary layers
Dynamically adapted mesh
Reaction-diffusion-advection equation
Coefficient inverse problem
Final time observed data

ABSTRACT

We propose the numerical method for solving coefficient inverse problem for a nonlinear singularly perturbed reaction-diffusion-advection equation with the final time observation data based on the asymptotic analysis and the gradient method. Asymptotic analysis allows us to extract *a priory* information about interior layer (moving front), which appears in the direct problem, and boundary layers, which appear in the conjugate problem. We describe and implement the method of constructing a dynamically adapted mesh based on this *a priory* information. The dynamically adapted mesh significantly reduces the complexity of the numerical calculations and improve the numerical stability in comparison with the usual approaches. Numerical example shows the effectiveness of the proposed method.

1. Introduction

In this paper the numerical method for solving the coefficient inverse problem for singularly perturbed reaction-diffusion-advection (RDA) equations with the final time observation data is developed. The method is based on the asymptotic analysis and optimization method. Due to the nonlinearity of the considered direct and inverse problems we minimize the cost functional by the gradient method to find approximate solution of the coefficient inverse problem. On each step of the gradient method the direct and corresponding conjugate problems are solved. The statements of these problems have the small parameter at higher derivative. Therefore the solutions of direct and conjugate RDA problems often feature narrow

☆ The work was supported by RFBR (projects NNo. 17-01-00519, 17-01-00670, 17-01-00159, 16-01-00755, 16-29-15120 and 16-01-00437), International Mathematical Center of Novosibirsk State University and the Ministry of Education and Science of the Russian Federation.

* Corresponding author.
E-mail addresses: lukyanenko@physics.msu.ru (D.V. Lukyanenko), mshishlenin@ngs.ru (M.A. Shishlenin), volkovvt@mail.ru (V.T. Volkov).

http://dx.doi.org/10.1016/j.cnsns.2017.06.002
1007-5704/

boundary and interior layers (stationary or moving fronts) and are extremely difficult for a numerical treatment. The main idea of our approach is that the asymptotic analysis of the direct problems allows to reduce significantly the complexity and optimize the numerical calculations, save the computational resources and improve the numerical stability of solving of corresponding singularly perturbed inverse problems.

The asymptotic analysis allows us to locate boundary and interior layers and to develop an efficient algorithm due to the following reasons: the smaller the parameter, the more rough and unstable numerical solution we obtain and more precise *a priori* information about the exact solution we get from asymptotic analysis. This fact gives the possibility for a productive combination of asymptotic and numerical approaches.

Asymptotic analysis also allows to prove the existence of the solution of certain class for the direct problem and highlight *a priori* information for solving of the inverse problem. Moreover, if consider the solutions with the internal layers or moving fronts, the spatial dimension of some problems for numerical calculations can be decreased. For example, to obtain *a priori* information about the location or the speed of the internal layer (moving front) we solve the equations with less spatial dimension than the original partial differential equation (PDE) problem. Namely, if the original PDE problem is n–dimensional, the equation for the moving front location is $(n-1)$–dimensional (this equation can be even ordinary differential or algebraic).

We demonstrate our approach on solving of initial-boundary problem for the following singularly perturbed reaction-diffusion-advection equation ($\varepsilon \ll 1$):

$$\varepsilon \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + q(x)\,u. \tag{1.1}$$

This type of equation or systems are used in mathematical models of biology, chemical kinetics, theory of combustion e.t.c. The asymptotic analysis of the direct problem for Eq. (1.1) has been already performed in [1,2]. Later it was applied to construct the dynamically adapted mesh based on the *a priory* information about the solution [3].

Different types of using *a priori* information in coefficient inverse problems were considered in [4–6] but in this paper we represent completely new approach that allows to solve inverse problems for singularly perturbed equations more effectively.

The paper is structured as follows. In Section 2 we discuss the general statement of the coefficient inverse problem which solution usually can not have any special features, however during the solution of relevant direct and conjugate problems, the difficulties connected with the interior and boundary layers may arise. Then, in Section 3, we formally mention the general method of its solution. In Section 4 we describe the main ideas of asymptotic theory that allows to get *a priori* information for the dynamically adapted mesh construction. In Section 5 we explain some nuances of the dynamically adapted mesh constructing. In Section 6 we perform a numerical experiment for some particular example to demonstrate the effectiveness of the proposed asymptotic-numerical approach.

2. Problem formulation

Let us consider the direct problem

$$\begin{cases} \varepsilon \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial t} = -u\dfrac{\partial u}{\partial x} + q(x)\,u, \quad x \in (0,1), \quad t \in (0,T), \\ u(0,t) = u_l(t), \quad u(1,t) = u_r(t), \qquad t \in (0,T), \\ u(x,0) = u_0(x), \qquad x \in (0,1). \end{cases} \tag{2.1}$$

Inverse problem consists of finding function $q(x)$ by known additional information:

$$u(x,T) = f_{obs}(x), \qquad x \in (0,1). \tag{2.2}$$

We apply the optimal control problem for the numerical solution (2.1)–(2.2).

Ill-posedness analysis

Let us show that the inverse problem (2.1)–(2.2) is ill-posed [7]. We will analyse two formulations of inverse problem: linear and nonlinear.

Let us investigate ill-posedness of the problem of recovering the initial data by known final time measured data (reverse time problem). We will construct the example of instability. The problem of recovering the initial data is more "simple" in comparison with the original problem formulation. Therefore, if we investigate the ill-posedness of reverse time problem, we can expect that the instability effects will influence harder in the numerical algorithm of the solution of the nonlinear coefficient inverse problem.

The linear analog of the problem (2.1) is:

$$\begin{cases} \varepsilon \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial t} = -\dfrac{\partial u}{\partial x}, \qquad x \in (0,1), \quad t \in (0,T), \\ u(0,t) = 0, \quad u(1,t) = 0, \qquad t \in (0,T), \\ u(x,0) = q(x), \qquad x \in (0,1). \end{cases} \tag{2.3}$$