Commun Nonlinear Sci Numer Simulat 55 (2018) 237–247

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Non-dispersive conservative regularisation of nonlinear shallow water (and isentropic Euler equations)

Didier Clamond[a,*], Denys Dutykh[b]

[a] Université Côte d'Azur, Laboratoire J. A. Dieudonné, CNRS UMR 7351, Parc Valrose, F-06108 Nice cedex 2, France
[b] LAMA, UMR 5127 CNRS, Université Savoie Mont Blanc, Campus Scientifique, F-73376 Le Bourget-du-Lac Cedex, France

ARTICLE INFO

Article history:
Received 16 April 2017
Revised 6 July 2017
Accepted 12 July 2017
Available online 19 July 2017

Keywords:
Shallow water flows
Dispersionless
Conservative
Regularisation

ABSTRACT

A new regularisation of the shallow water (and isentropic Euler) equations is proposed. The regularised equations are non-dissipative, non-dispersive and posses a variational structure; thus, the mass, the momentum and the energy are conserved. Hence, for instance, regularised hydraulic jumps are smooth and non-oscillatory. Another particularly interesting feature of this regularisation is that smoothed 'shocks' propagates at exactly the same speed as the original discontinuous ones. The performance of the new model is illustrated numerically on some dam-break test cases, which are classical in the hyperbolic realm.

1. Introduction

In fluid mechanics, many phenomena can be described by hyperbolic partial differential equations, such as the inviscid Burgers equation [1], the isentropic Euler equations [2] and the shallow water (Airy or Saint-Venant [3]) equations. The latter, for flat seabeds in one horizontal dimension, are most often written as mass and momentum flux conservations

$$h_t \;+\; \partial_x[\,h\,u\,] \;=\; 0, \tag{1}$$

$$\partial_t[\,h\,u\,] +\; \partial_x\Big[\,h\,u^2 \;+\; \tfrac{1}{2}\,g\,h^2\,\Big] \;=\; 0, \tag{2}$$

where $u = u(x,t)$ is the depth-averaged horizontal velocity (x the horizontal coordinate, t the time), $h = d + \eta(x,t)$ is the total water depth (d the mean depth, η the surface elevation from rest) and g is the (downward) acceleration due to gravity (see the sketch in Fig. 1). These equations describe nonlinear non-dispersive long surface gravity waves propagating in shallow water. They are equations of choice when one is interested in modelling large scale phenomena without resolving the details at the small scales, for instance, in tsunamis and tides modelling. It should be noted that Eqs. (1) and (2) are mathematically identical to the isentropic Euler equations with $\gamma = 2$ (γ the ratio of specific heats) describing some compressible fluids [4]. Here, we focus on shallow water equations, but it is clear that our claims apply as well to the isentropic Euler and mathematically similar equations.

Hyperbolicity is a nice feature of Eqs. (1) and (2) because they can be tackled analytically and numerically with powerful methods (e.g., characteristics, finite volumes, discontinuous Galerkin). A major inconvenient is that these equations admit

* Corresponding author.
E-mail addresses: diderc@unice.fr (D. Clamond), Denys.Dutykh@univ-savoie.fr (D. Dutykh).
URL: http://math.unice.fr/~didierc/ (D. Clamond), http://www.denys-dutykh.com/ (D. Dutykh)

http://dx.doi.org/10.1016/j.cnsns.2017.07.011
1007-5704/

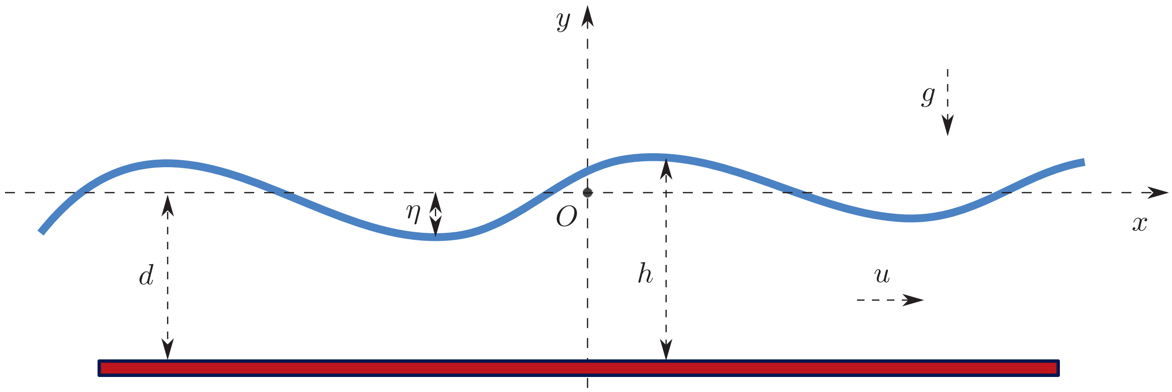

Fig. 1. Sketch of the fluid domain.

non-unique weak solutions and entropy considerations have been proposed to ensure unicity [5]. In gas dynamics, these weak solutions correspond to shock waves and the loss of regularity can be problematic, in particular for computations using spectral methods (even if some spectral approaches have been developed for hyperbolic equations as well [6]). Several methods have then been introduced to regularise the equations and, in particular, to avoid the formation of sharp discontinuous shocks (replacing them by smooth tanh -like profiles). Perhaps, the first regularisation was proposed by J. Leray [7] in the context of incompressible Navier–Stokes equations. His theoretical programme consisted in showing the existence of solutions in regularised equations, subsequently taking the limit $\epsilon \to 0$ (ϵ a small regularising parameter) to obtain weak solutions of Navier–Stokes.

A method of regularisation consists in first adding an artificial viscosity into the equations, and in taking the limit of vanishing viscosity in a second time. This method was introduced by von Neumann and Richtmyer [8]. It allows to generalise the classical concept of a solution and to prove eventually the uniqueness, existence and stability results for viscous regularised solutions [9,10]. Due to the added viscosity, the energy is no longer conserved, that can be a serious drawback for some applications, for instance for long time simulations when the shocks represent (unresolved) small scale phenomena that are not dissipative.

Another regularisation consists in adding weak dispersive effects to the equations [11]. As shown by Lax [12], the dispersive regularisation is not always sufficient to obtain a reasonable limit to weak entropy solutions as the dispersion vanishes. Consequently, the most successful approach to study non-classical shock waves is to consider the combined dispersive-diffusive approximations [13]. Also, the added dispersion can generate high-frequency oscillations that must be resolved by the numerical scheme, resulting in a significant increase in the computational time. Nonlinear diffusive–dispersive regularisations for the scalar case were considered in [14]. The main goal was to obtain a regularised model which admits the existence of classical solutions globally in time.

Yet another, less known, regularisation inspired by Leray's method [7], consists filtered the velocity field such that the resulting equations are non-dissipative and non-dispersive. Such regularisations have been proposed for the Burgers [15], for isentropic Euler [16–18] and other [19] equations. In the literature, this regularisation method appears with various denominations, such as *Leray-type regularisation*, *α-regularisation* and *Helmholtz regularisation*. A drawback of this method is that the regularised (then smooth) shocks propagate at a speed different than that of the original equations. This drawback, among other things, is addressed in the present paper.

In this paper, we propose a new type of regularisation which is both non-dissipative and non-dispersive. This regularisation preserves the conservations of mass, momentum and energy, that is an important feature for some physical applications, in particular for long time simulations. The derivation proposed below is based on a recent work [20] where a Lagrangian, suitable for long waves propagating in shallow water, was modified to incorporate one free parameter that can be used to improve the dispersion properties. Here, we make another step introducing two independent parameters. But, instead of improving the dispersion, these parameters are chosen to cancel the dispersion and thus to provide a regularisation of the classical shallow water (Saint-Venant) equations. In addition to being conservative and non-dispersive, this model yields regularised (smooth) 'shocks' that propagate exactly at the same speed as in the original model. The properties of the obtained model are discussed below, mostly via numerical evidences.

The present manuscript is organised as follows. In Section 2, a new two-parameter generalised shallow water model is introduced. In Section 3, the two parameters are related in a way to provide a non-dispersive non-diffusive and conservative regularised shallow water equations. The jump conditions of Rankine–Hugoniot type on both sides of a shock wave are discussed in Section 3.4. These equations admit regular travelling wave solutions, as shown in Section 3.3. In Section 4, several numerical examples are provided, demonstrating the efficiency of the method. Finally, some conclusions and perspectives are outlined in Section 5.