Computers and Fluids 156 (2017) 441–448

Contents lists available at ScienceDirect

Computers and Fluids

journal homepage: www.elsevier.com/locate/compfluid

Technical note

Extension of a well-balanced central upwind scheme for variable density shallow water flow equations on triangular grids

Sepideh Khorshid*, Abdolmajid Mohammadian, Ioan Nistor

Department of Civil Engineering, University of Ottawa, 161 Louis Pasteur, Ottawa, Ontario K1N 6N5, Canada

ARTICLE INFO

Article history:
Received 25 December 2016
Revised 6 June 2017
Accepted 4 August 2017
Available online 5 August 2017

Keywords:
Shallow water equations
Variable density
Well-balanced central upwind scheme
Two-dimensional depth-averaged model

ABSTRACT

In this paper, the central upwind scheme for variable density shallow water system of equations is extended to triangular discretization of the domain. In this scheme, the well-balanced and positivity preserving properties are maintained such that the large oscillations and noises are avoided in the solution. Furthermore, time-history of flow surface always remains non-negative throughout the simulations. Various properties of the scheme are validated using several benchmark data. Also, the accuracy and efficiency of the methodology are tested by comparing the results of the model to other complex scheme for some test cases. The method ensures high computational efficiency while maintaining the accuracy of the results and preserves two types of "lake at rest" steady states, and is oscillation free across the small density change.

1. Introduction

Environmental free surface shallow water flow has numerous applications for the transport debris and suspended sediment especially under slope conditions. When such flows enter larger rivers or other types of water bodies such as lakes, the significant difference between the densities of the two systems need to be accounted for in order to ensure accurate simulation results. As such, domestic and industrial effluents from outfall structures often have a different density than that of the ambient water body, something which leads to various flow and mixing characteristics of the discharge.

To model the mixing of the two interacting water bodies, three-dimensional models such as OpenFOAM, Delft3D and MIKE-3 have been increasingly used. However, due to their high computational costs, if scientifically justifiable, it is more efficient to use two-dimensional depth-averaged models because of their simplicity in implementation and application, especially in the initial stages of the design [7]. Therefore, in many cases if the ambient stream can be approximated to a shallow stream, the use of shallow water equations can lead to some of the most performant tools in modeling mixing problems. To use the traditional shallow water equations in modeling mixing, it is necessary to further modify them such as to include the option of density change, which has been the focus of many previous researches using shallow water equations.

Shallow water flows have been widely studied. Abbott [1] and Weiyan [12] discussed numerical aspect of the shallow water type equations and provided a systematic account of the principles of computational hydraulics and their application to free surface flows. Those studies did not consider the cases with abrupt change in underwater topography, and the possibility of mixing of flows with different densities. Sleight et al. [10] refined the method to be able to handle complex flow domains focusing only on the conservative form of the shallow water equations.

Brice et al. [3] treated dissolved salt in water as a variable in the momentum conservation by applying the Boussinesq approximation to preserve the density effects in the hydrostatic pressure term. The suggested formulation allowed a robust simulation of the horizontal flow dynamics; while preserving the efficiency of a characteristic method.

Solving the shallow water system is a challenging task, which needs a robust numerical method. The method should be well-balanced [2], which means that it should exactly preserve the steady-state solutions. If the method does not accurately respect the balance, the numerical method may lead to significant oscillations, even overshadowing the main stream [4,5]. Moreover, the numerical method should be positivity preserving, which means

* Corresponding author.
E-mail addresses: skhor086@uottowa.ca (S. Khorshid), majid.mohammadian@uottawa.ca (A. Mohammadian), inistor@uottawa.ca (I. Nistor).

http://dx.doi.org/10.1016/j.compfluid.2017.08.005
0045-7930/

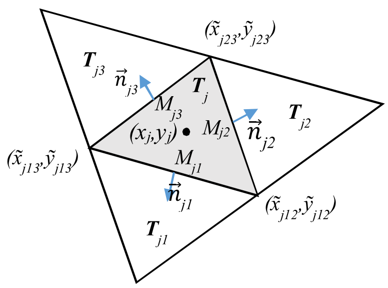

Fig. 1. Triangular cell used in the proposed model.

that in order to evaluate the eigenvalues of the system, the water level above the bed should remain positive at all times. This property is crucial where in parts of the analytical domain no water is present, or when the depth of water is very small, and small oscillations may lead to negative depths, eventually resulting in the simulation to fail.

In this study, the central upwind scheme developed in Bryson et al. [4] is applied and extended to the variable density shallow water equations following the work of Brice et al. [3]. Due to the use of the Bousinesq approximation in the formulation suggested by Brice et al. [3], it is assumed that the density gradients are small. Therefore, the methodology used here is more efficient compared to the similar works by other investigators, when the concentration of the sediment in suspension is not very high. This scheme considers the well-balanced and positivity-preserving characteristics of the dense flows. In this regard, the analytical solution is presented over a triangular discretization of the domain, using a higher order temporal and spatial numerical scheme which is discussed in detail.

Following this introduction, Section 2 of the paper describes the application of the numerical model, as well as discretization of the source terms required to reach the well-balanced property. Numerical examples are further presented in Section 3 in order to validate different aspects of the proposed solution. Some concluding remarks complete this study.

2. Governing equations

In the current study, the two-dimensional (2D) Saint-Venant system of shallow water equations as presented below are used.

$$\boldsymbol{U}_t + [\boldsymbol{F}(U)]_x + [\boldsymbol{G}(\boldsymbol{U})]_y = \boldsymbol{S}(\boldsymbol{U}) \tag{1}$$

in which, vectors **U, F, G**, and **H** are defined as

$$\boldsymbol{U} = \begin{pmatrix} h \\ uh \\ vh \\ \rho h \end{pmatrix}, \quad \boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} uh \\ u^2 h + \dfrac{g}{2\rho_0}\rho h^2 \\ uvh \\ \rho uh \end{pmatrix},$$

$$\boldsymbol{G}(\boldsymbol{U}) = \begin{pmatrix} vh \\ uvh \\ v^2 h + \dfrac{g}{2\rho_0}\rho h^2 \\ \rho vh \end{pmatrix}, \quad \boldsymbol{S}(\boldsymbol{U}) = \begin{pmatrix} 0 \\ -\rho g h B_x - \tau_{b,x} \\ -\rho g h B_y - \tau_{b,y} \\ 0 \end{pmatrix} \tag{2}$$

where, h is the water depth above the bed elevation B, u and v are the depth-averaged velocities in x- and y-direction, respectively, ρ is the average density of the fluid, g is the acceleration of gravity, ρ_0 is the reference freshwater density and $\tau_{b,\,x}$ and $\tau_{b,\,y}$ are the bed shear stress values in x- and y- direction, respectively. Indices of x, y, and t represent the first derivatives of the parameters with respect to the spatial 2D and time coordinates, respectively.

One of the objectives of this study is to extend the well-balanced property for the variable density shallow water equations. Therefore, one can first rewrite the system in an equivalent form in terms of $\omega := h + B,\ q := uh,\ p := vh,\ r := \rho h$ and neglecting the effects of the bottom friction terms $(\tau_{bx},\ \tau_{by})$ as

$$\begin{aligned} &\omega_t + q_x + p_y = 0 \\ &q_t + \left[\frac{q^2}{\omega - B} + \frac{gr}{2\rho_0}(\omega - B)\right]_x + \left[\frac{pq}{(\omega - B)}\right]_y = -rgB_x/\rho_0 \\ &p_t + \left[\frac{pq}{(\omega - B)}\right]_x + \left[\frac{p^2}{\omega - B} + \frac{gr}{2\rho_0}(\omega - B)\right]_y = -rgB_y/\rho_0 \\ &r_t + \left[\frac{rq}{(\omega - B)}\right]_x + \left[\frac{rp}{(\omega - B)}\right]_y = 0 \end{aligned} \tag{3}$$

Kurganov and Petrova [8] introduced a new central-upwind scheme on general triangular grids to solve the two-dimensional conservation laws. Unlike Godunov-type schemes, this can be applied to complex geometries while maintaining its simplicity and robustness. In the present study, the triangular domain of $\tau := \bigcup_j T_j (T_j$ being the triangular cells of size $|T_j|)$ is used. This discretization of this domain, is shown in Fig. 1, where $\vec{n}_{jk} := (\cos(\theta_{jk}), \sin(\theta_{jk}))$ is the outer unit vector, normal to the corresponding sides of T_j with the length l_{jk} $(k = 1,2,3)$. x_j, and y_j represent the coordinates of the mass center of T_j, while $M_{jk} = (x_{jk}, y_{jk})$ is the midpoint of the k-th side of the T_j triangle. In this study, the authors applied the central upwind scheme proposed by Kurganov

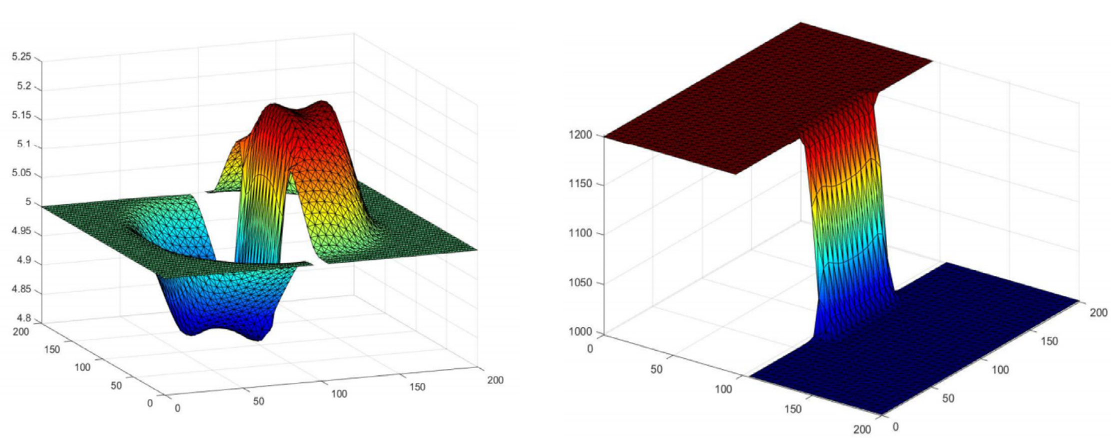

Fig. 2. (a) Water surface elevation, and (b) density distribution 5.2 s after the dam break initiation.