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On the analysis of pure bending of rigid-plastic beams in strain-gradient plasticity

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ABSTRACT

The complete stress field, including the microstress, the moment-stress, and the line forces are derived for the pure bending of a rigid-plastic beam of rectangular cross-section in the model of strain-gradient plasticity. The workless spherical parts of the microstress and the moment-stress tensors are incorporated in the analysis. Their determination is shown to be of importance for the fulfilment of the higher-order traction boundary conditions, the physical interpretation of line forces, and their contributions to bending moments. Three equivalent methods are used to derive the moment-curvature relationship for any of the gradient-enhanced effective plastic strain measures from the considered broad class of these measures. Specific results are given for the selected choice of the stress-strain relationship describing the uniaxial tension test. Closed-form analytical expressions are obtained in the case of linear hardening, and in some cases of nonlinear hardening. The analysis of the plane-strain bending of thin foils is also presented. In this case there are two sets of line forces along the edges of the beam. The relationships between the applied bending moment and the curvature, and between the lateral bending moment and the curvature are derived and discussed. The lateral bending moment along the lateral sides of the beam, needed to keep the plane-strain mode of deformation, is one-half of the applied bending moment.

1. Introduction

In the strain-gradient plasticity theory the microstress and the moment-stress are the deviatoric tensors introduced as the work-conjugates to the deviatoric plastic strain tensor and its gradient (Fleck and Hutchinson, 1997, 2001; Gudmundson, 2004; Anand et al., 2005; Gurtin and Anand, 2005, 2009; Bardella, 2006; Fleck and Willis, 2009; Hutchinson, 2012; Nielsen and Niordson, 2014; Fleck et al., 2014, 2015; Lubarda, 2016a). We show in this paper through the analysis of pure bending of a rigid-plastic beam that it is important to incorporate in the analysis the (workless) spherical parts of the microstress and the moment-stress tensors, in addition to their deviatoric parts. This allows the fulfilment of the higher-order traction boundary conditions, the interpretation of the line forces along the intersection of the bounding surfaces of the beam, and the evaluation of the (reactive) lateral bending moment in the case of plane-strain bending. In Sections 2–4 we present a brief review of the J_2 deformation theory of strain-gradient plasticity by using an arbitrary measure of the gradient-enhanced effective plastic strain from a wide class of these measures introduced in the literature (Fleck and Hutchinson, 1997, 2001). The stress fields in a rigid-plastic beam of rectangular cross-section under pure bending are derived in Section 5. The deviatoric pats are determined by usual means from the established constitutive equations, while the spherical parts are determined by the fulfilment of the higher-order traction boundary conditions. Three equivalent methods are used in Section 6 to derive the moment-curvature relationship for any of the utilized gradient-enhanced effective plastic strain measures. Closed-form analytical expressions are obtained in the case of linear hardening, and for some measures of the effective plastic strain in the case of nonlinear hardening. For an adopted nonlinear stress-strain relationship describing the uniaxial tension test, the moment-curvature relationships are evaluated numerically in Section 7. The analysis of the plane-strain model of the bending of a wide rigid-plastic beam is presented in Section 8, since such model has been commonly adopted in the bending analysis of thin foils (Stölken and Evans, 1998; Huang et al., 2000; Wang et al., 2003; Haque and Saif, 2003; Voyiadjis and Abu Al-Rub, 2005; Engelen et al., 2006; Lou et al., 2006; Evans and Hutchinson, 2009; Idiart

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et al., 2009). Two types of line forces are shown to act along different edges of the beam. They are used to derive the applied bending moment-curvature and the lateral bending moment-curvature relationships. The lateral stresses are shown to be equal to one-half of the corresponding longitudinal stresses. As a consequence, the lateral bending moment (per unit length of the beam) is equal to one-half of the applied bending moment (per unit width of the beam).

2. Gradient-enhanced effective plastic strain

In a simple formulation of the deformation theory of strain-gradient plasticity (Hutchinson, 2012), the specific plastic work (per unit volume) is expressed in terms of the gradient-enhanced effective plastic strain E_{p} by

$$w_{\mathrm{p}}(E_{\mathrm{p}}) = \int_0^{E_{\mathrm{p}}} \sigma_0(\varepsilon_{\mathrm{p}})\,\mathrm{d}\varepsilon_{\mathrm{p}}, \tag{1}$$

where $\sigma_0 = \sigma_0(\varepsilon_{\mathrm{p}})$ represents the stress-strain curve from the uniaxial tension test. A wide class of the gradient-enhanced effective plastic strain measures, each involving one (albeit possibly different) material length parameter l (Fleck and Hutchinson, 1997; Evans and Hutchinson, 2009), is

$$E_{\mathrm{p}} = \left(e_{\mathrm{p}}^s + l^s g_{\mathrm{p}}^s\right)^{1/s}, \quad (s \geq 1)\,. \tag{2}$$

Here, e_{p} is the effective plastic strain and g_{p} the effective plastic strain-gradient, defined by

$$e_{\mathrm{p}} = \left(\frac{2}{3}\varepsilon_{ij}^{\mathrm{p}}\varepsilon_{ij}^{\mathrm{p}}\right)^{1/2}, \quad g_{\mathrm{p}} = \left(\frac{2}{3}\varepsilon_{ij,k}^{\mathrm{p}}\varepsilon_{ij,k}^{\mathrm{p}}\right)^{1/2}. \tag{3}$$

The two most frequently used measures are associated with the choices $s = 1$ and $s = 2$, which specify E_{p} as either a linear or harmonic sum of e_{p} and lg_{p}, i.e.,

$$E_{\mathrm{p}} = e_{\mathrm{p}} + lg_{\mathrm{p}}\,, \quad E_{\mathrm{p}} = \left(e_{\mathrm{p}}^2 + l^2 g_{\mathrm{p}}^2\right)^{1/2}. \tag{4}$$

The plastic strain is taken to be

$$\varepsilon_{ij}^{\mathrm{p}} = e_{\mathrm{p}} m_{ij}\,, \quad m_{ij} = \frac{3}{2}\frac{\sigma'_{ij}}{\sigma_{\mathrm{eq}}}\,, \tag{5}$$

with the equivalent stress

$$\sigma_{\mathrm{eq}} = \left(\frac{3}{2}\sigma'_{ij}\sigma'_{ij}\right)^{1/2}, \tag{6}$$

where prime designates the deviatoric part.

The total infinitesimal strain ε_{ij} is the sum of elastic and plastic contributions. For the rigid-plastic material model of concern in this paper, the elastic component is absent, so that $\varepsilon_{ij} = \varepsilon_{ij}^{\mathrm{p}}$.

3. Work-conjugates to plastic strain and its gradient

It is assumed that the plastic strain-gradients $\varepsilon_{ij,k}^{\mathrm{p}}$ contribute to the work per unit volume. The work conjugate to $\varepsilon_{ij,k}^{\mathrm{p}}$ is the moment-stress $\tau_{ijk} = \tau_{jik}$, and the work-conjugate to $\varepsilon_{ij}^{\mathrm{p}}$ is the microstress $q_{ij} = q_{ji}$, such that (Gudmundson, 2004)

$$\dot{w}_{\mathrm{p}} = q_{ij}\dot{\varepsilon}_{ij}^{\mathrm{p}} + \tau_{ijk}\dot{\varepsilon}_{ij,k}^{\mathrm{p}}\,. \tag{7}$$

Since $\varepsilon_{kk}^{\mathrm{p}} = 0$, only the deviatoric parts of q_{ij} and τ_{ijk} contribute to plastic work. To identify them, we evaluate the rate of plastic work from (1) as

$$\dot{w}_{\mathrm{p}} = \sigma_0(E_{\mathrm{p}})\dot{E}_{\mathrm{p}}\,. \tag{8}$$

By the differentiation of (2) and (3), the rate of the gradient-enhanced effective plastic strain is found to be

$$\dot{E}_{\mathrm{p}} = \frac{2}{3}\,E_{\mathrm{p}}^{1-s}\left(e_{\mathrm{p}}^{s-2}\varepsilon_{ij}^{\mathrm{p}}\dot{\varepsilon}_{ij}^{\mathrm{p}} + l^s g_{\mathrm{p}}^{s-2}\varepsilon_{ij,k}^{\mathrm{p}}\dot{\varepsilon}_{ij,k}^{\mathrm{p}}\right). \tag{9}$$

The substitution of (9) into (8), and the comparison with (7), establishes the constitutive expressions for the deviatoric parts

$$q'_{ij} = \frac{2}{3}\,\frac{\sigma_0(E_{\mathrm{p}})}{E_{\mathrm{p}}^{s-1}}\,e_{\mathrm{p}}^{s-2}\varepsilon_{ij}^{\mathrm{p}}\,, \quad \tau'_{ijk} = \frac{2}{3}l^s\frac{\sigma_0(E_{\mathrm{p}})}{E_{\mathrm{p}}^{s-1}}\,g_{\mathrm{p}}^{s-2}\varepsilon_{ij,k}^{\mathrm{p}}\,. \tag{10}$$

The spherical parts of q_{ij} and τ_{ijk} do not contributing to the plastic work, and at this point of the analysis are arbitrary. If needed, however, they may be specified by the consideration of the higher-order traction boundary conditions, as discussed in Section 5.1 in the context of the rigid-plastic beam bending.

4. Principle of virtual work

In the case of a rigid-plastic body of volume V, bounded by a piece-wise smooth surface S, the principle of virtual work is

$$\int_V \left(q'_{ij}\delta\varepsilon_{ij}^{\mathrm{p}} + \tau'_{ijk}\delta\varepsilon_{ij,k}^{\mathrm{p}} + \frac{1}{3}\,\sigma_{ii}\delta\varepsilon_{jj}^{\mathrm{p}}\right)\mathrm{d}V = \int_S \left[\widehat{T}_i\delta u_i + \widehat{R}_i D(\delta u_i)\right]\mathrm{d}S + \sum_n \oint_{C_n} p_i\delta u_i\,\mathrm{d}C_n\,, \tag{11}$$

provided that the equations of equilibrium hold

$$\sigma_{ij,j} = 0\,, \quad \tau_{ijk,k} + \sigma_{ij} - q_{ij} = 0\,, \tag{12}$$

together with the relations

$$T_i = \sigma_{ij}n_j\,, \quad t_{ij} = \tau'_{ijk}n_k \tag{13}$$

between the traction vector T_i and the Cauchy stress tensor σ_{ij}, and between the (deviatoric) moment-traction tensor t_{ij} are the deviatoric part of the moment-stress tensor τ'_{ijk}. The components of the outward unit vector, orthogonal to the considered surface element, are denoted by n_i, and u_i are the displacement components.

The three independent traction components $\widehat{T}_i$ are

$$\widehat{T}_i = \overline{T}_i - n_i n_j R_j(D_k n_k) - D_i(n_j R_j)\,, \tag{14}$$

with

$$\overline{T}_i = T_i + R_i(D_j n_j) - D_j t_{ij}\,, \quad T_i = \sigma_{ij}n_j\,, \tag{15}$$

while the two independent higher-order traction components $\widehat{R}_i$, tangential to S, are

$$\widehat{R}_i = R_i - n_i n_j R_j\,, \quad R_i = t_{ij}n_j\,, \tag{16}$$

with $t_{ij} = \tau'_{ijk}n_j n_k$. The utilized surface gradient operator is defined