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Diffusion-induced bending of viscoelastic beams

Fuqian Yang

Department of Chemical and Materials Engineering, University of Kentucky, Lexington, KY 40506, USA

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ABSTRACT

The contribution of local volumetric change due to the diffusion/migration of solute atoms to viscoelastic deformation is incorporated in the theory of linear viscoelasticity, following the elastic theory of diffusion-induced stress. Three-dimensional constitutive relationship in differential form for diffusion-induced stress in linear viscoelastic materials is proposed. Using the correspondence principle between linear viscoelasticity and linear elasticity and the results from the diffusion-induced bending of elastic beams, the radii of curvature of the centroidal plane of viscoelastic beams of single layer and bilayer with top layer being viscoelastic in the transform domain are obtained. For viscoelastic beams of single layer, closed-form solution of the radius of curvature of the centroidal plane is derived, and the radius of curvature is inversely proportional to the diffusion moment created by non-uniform distribution of solute atoms. For the condition of constant concentration on free surface, there is overshoot behavior; for the condition of constant flux on free surface, there is no overshoot behavior. For viscoelastic beams of bilayer with top layer being the Maxwell-type standard material, the numerical results show the presence of the overshoot behavior for a very compliant elastic layer under the condition of constant concentration on free surface, and there is no overshoot behavior under the condition of constant flux on free surface.

1. Introduction

The progress in the micro- and nanomanufacturing technologies has made it possible to manufacture cantilever-based structures for sensing techniques [1–5]. Similar structures, beam-based structures, have also been used to analyze the change of surface stress due to electrochemical charging-discharging [6–10] and the stresses induced by mass transport [7,11–15]. In the heart of the cantilever-based sensing techniques is the change of surface stress associated with adsorption of molecules on "active" surface/coating and/or the volumetric strain associated with phase change and/or mass transport, which can cause the deflection of the cantilever-based structures.

Currently, the deflection analysis of cantilever-based structures and beam-based structures has mainly based on the theory of elastic beams with the incorporation of surface stress and the strain due to swelling/shrinking of coated materials. It is known that polymer coatings have been used in cantilever-based chemical sensors [9,12,15]. The theory of elastic beams likely cannot reveal the temporal evolution of the beam deflection with a coating of polymer. For example, Pei and Inganäs [13] used the theory of linear elasticity to model the deflection of a bipolymer strip induced by cation insertion and salt draining, and stated that their numerical results may not compare with the experimental results of the increasing part of the bipolymer strip deflection. They suggested that this is likely due to the salt draining occurring immediately after the reduction. Observing an overshoot that slowly decreases to the steady-state value for the absorption of a chemical analyte into a polymer coating, Wenzel et al. [16] developed a model of absorption-induced static bending of a microcantilever coated with a viscoelastic material, and were able to demonstrate the overshoot behavior from their model. It is worth mentioning that Yang and Li [17] used the elastic theory of diffusion-induced stress to analyze the cantilever-based hydrogen sensor, and their results showed the overshoot behavior. Yang [18] used the theory of surface rheology to analyze the effect of a surface viscous film on the vibration of an elastic microcantilever. Approximating uniform temperature in a bilayer system, Hsueh et al. [19] used the correspondence principle to analyze the stress evolution in the bilayer consisting of Maxwell materials due to thermal and/or lattice mismatch. They did not analyze the temporal evolution of the radius of curvature of the viscoelastic bilayer system. There are little studies focusing on the diffusion-induced bending of cantilever-based structures consisting of viscoelastic materials.

It is known that polymer coatings have been widely used in cantilever-based sensing structures and conducting polymer as well as porous materials has been used in energy storage such as supercapacitors [20–24]. There exists local volumetric change associated with mass transport and phase transform in polymer and porous materials, and it needs to carefully study the effect of the volumetric strain on the structural sensetivity for the applications in sensing technology and en-

E-mail address: fyang2@uky.edu

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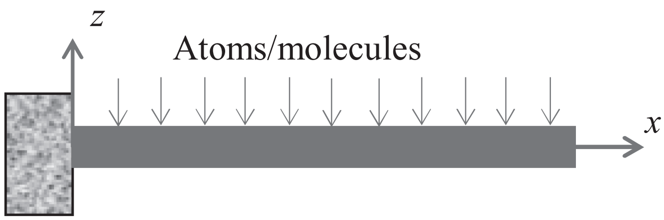

Fig. 1. Schematic diagram of the diffusion/migration of atoms/molecules into an elastic beam of single layer.

ergy storage. Considering the viscoelastic characteristcs of polymer and porou materials associated with fluid-structure interaction, the effect of the volumetric strain due to mass transport on the bending of viscoelastic beams is analyzed. The focuse is on the temporal evolution of the beam deflection. The theory of diffusion-induced stress in linear elasticity is extended to linear viscoelasticity.

2. Analyses of diffusion-induced bending of elastic beams

In the framework of linear elasticity, the constitutive relationship describing the diffusion-induced deformation of elastic materials is [25]

$$\boldsymbol{\varepsilon} = \frac{1}{E}[(1+\nu)\boldsymbol{\sigma} - \nu\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I}] + \frac{c\Omega}{3}\mathbf{I} \tag{1}$$

where $\boldsymbol{\varepsilon}$ is strain tensor, $\boldsymbol{\sigma}$ is stress tensor, $\mathbf{I}$ is unit tensor, Ω is the molar volume of solute atoms (m^3/mol), c is the concentration (mol/m^3) of diffusing component (solute atoms), and E and ν are Young's modulus and Poisson's ratio of the material, respectively. The molar volume of Ω is assumed constant, independent of c. The relationship between the strain tensor and the displacement vector ($\boldsymbol{u}$) is

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\boldsymbol{u} + \boldsymbol{u}\nabla) \tag{2}$$

Fig. 1 shows an elastic beam with the dimensions in the y- and z-directions much smaller than that in the x-direction. Eq. (1) reduces to

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} + \frac{c\Omega}{3} \tag{3}$$

for the diffusion-induced bending of the elastic beam.

Elastic beam of single layer

For completeness, the diffusion-induced bending of an elastic beam of Young's modulus of E_1 with the dimensions in the y- and z-directions much smaller than that in the x-direction is first briefly analyzed. For detailed derivation, see the work by Yang and Li [17]. According to the Bernoulli–Euler assumption that planar sections perpendicular to the axis remain planar after bending, one can express axial displacement (x-direction), $u(x, z)$, as

$$u(x,z) = f_0(x) + zf_1(x) \tag{4}$$

From Eq. (4), the axial normal strain, ε_{xx}, and stress, σ_{xx}, can calculated as

$$\varepsilon_{xx}(x,z) = f_0'(x) + zf_1'(x) \tag{5}$$

$$\sigma_{xx}(x,z) = E_1\left[f_0'(x) + zf_1'(x) - \frac{1}{3}c\Omega\right] \tag{6}$$

Here, $f_0(x)$ and $f_1(x)$ are to be determined from the equilibrium equations, and the primes denote differentiation with respect to x.

Assuming that the characteristic time for diffusion/migration of solute atoms into the elastic beam is much larger than the characteristic time for the propagation of elastic wave, one can approximate the deflection of the elastic beam as quasi-static. Under the condition of quasi-static state, the equilibrium equations are

$$\int_{A_1}\sigma_{xx}(x,z)dA = 0 \text{ and } \int_{A_1}z\sigma_{xx}(x,z)dA = 0 \tag{7}$$

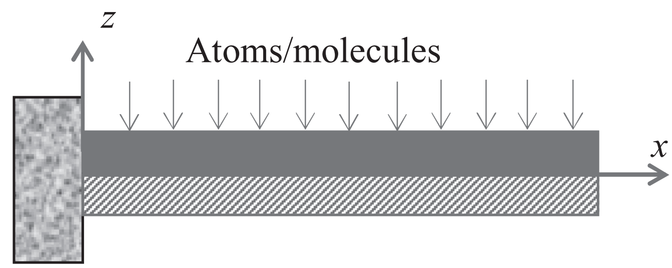

Fig. 2. Schematic diagram of the diffusion/migration of atoms/molecules into an elastic beam of bilayer.

where the integrations must cover the entire cross-sectional area of A_1.

Substituting Eq. (6) into Eq. (7) and using the condition of $\int_{A_1} zdA = 0$ (i.e. the centroidal plane is the middle plane of the elastic beam) yield

$$f_0'(x) = \frac{1}{3} < c > \Omega \text{ and } f_1'(x) = \frac{1}{3I_1}M_c\Omega \tag{8}$$

with

$$< c >= \frac{1}{A_1}\int_{A_1} c\,dA,\ M_c = \int_{A_1} czdA,\ \text{and } I_1 = \int_{A_1} z^2 dA \tag{9}$$

Here, M_c is defined as diffusion moment. Substituting Eq. (8) into Eqs. (5) and (6), one obtains the strain and stress in the elastic beam as

$$\varepsilon_{xx}(x,z) = \frac{1}{3} < c > \Omega + \frac{1}{3I_1}zM_c\Omega \tag{10}$$

$$\sigma_{xx}(x,z) = \frac{E_1\Omega}{3}\left[(< c > -c) + \frac{z}{I_1}M_c\right] \tag{11}$$

which gives the radius of curvature, ρ, of the centroidal plane as

$$\frac{1}{\rho} = -\frac{\partial\varepsilon_{xx}(x,z)}{\partial z} = -\frac{1}{3I_1}M_c\Omega \tag{12}$$

Elastic beam of bilayer

Consider an elastic beam consisting of two elastic layers with solute atoms only being able to migrate/diffuse into top layer, as shown in Fig. 2. There is perfect bonding between top layer and bottom layer, and there is no slip between these two layers. The elastic moduli are E_1 and E_2 for the top layer and the bottom layer, respectively.

Similar to the analysis of the bending of the elastic beam of single layer, the axial displacement (x-direction), $u(x, z)$, can be expressed as

$$u(x,z) = f_0(x) + zf_1(x) \tag{13}$$

in which the x-axis ($z = 0$) is located in the interface between the top layer and the bottom layer. The axial strains of each layer can then be calculated as

$$\frac{\sigma_{xx1}}{E_1} = \varepsilon_{xx1} - \frac{1}{3}c\Omega = f_0'(x) + zf_1'(x) - \frac{1}{3}c\Omega \qquad \text{for top layer} \tag{14}$$

$$\frac{\sigma_{xx}}{E_2} = \varepsilon_{xx2} = f_0'(x) + zf_1'(x) \qquad \text{for bottom layer} \tag{15}$$

Without the action of external loading, the equilibrium equations give

$$\int_{A_1}\sigma_{xx_1}(x,z)dA + \int_{A_2}\sigma_{xx_2}(x,z)dA = 0 \tag{16}$$

$$\int_{A_1}z\sigma_{xx_1}(x,z)dA + \int_{A_2}z\sigma_{xx_2}(x,z)dA = 0 \tag{17}$$

Here, A_1 and A_2 are the cross-sectional areas of the top layer and the bottom layer, respectively. Substituting Eqs. (14) and (15) into