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Discrete singular convolution and Taylor series expansion method for free vibration analysis of beams and rectangular plates with free boundaries

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ABSTRACT

Since the excellent performance of the discrete singular convolution (DSC) for high order frequency analysis has not been demonstrated for rectangular plates with free edges and corners thus far, this paper proposes a novel approach in applying the DSC for vibration analysis of beams and rectangular plates with arbitrary boundary conditions. The Taylor series expansion method is used to apply the boundary conditions. The existing difficulty encountered at plate corner points is overcome by the proposed method. The essential idea of the novel method is that two Taylor series expansions with different orders are used to eliminate the fictitious points outside the physical domain. Various beam and rectangular plate problems are solved by the proposed DSC and the results are compared with exact solutions, finite element data and results obtained by the differential quadrature method (DQM) using much higher number of grid points. It is demonstrated that the Taylor series expansion method to apply the boundary conditions is general and accurate. It is also found that the performance of the proposed DSC is excellent and independent of boundary conditions.

1. Introduction

Beams and plates are the basic structural elements in engineering structures and their static, buckling and vibration behaviors are of important to the designers. Therefore, beams and plates have been received great attention. Since analytical solutions are only available for certain simple cases [1], numerical method such as the finite element method (FEM) is the major approach in engineering practice.

However, certain limitations exist in the conventional FEM. Thus other computationally efficient numerical methods are being sought. The research into the development of new methods for numerical solution of problems in engineering and physical sciences remains motivated by the needs of modern technology [2]. For the vibration analysis of beams and plates, the discrete singular convolution (DSC), proposed by Wei [3], is a very promising approach [4,5]. The mathematical foundation of the DSC is the theory of distributions and the theory of wavelets. The DSC algorithm has been realized in both collocation and Galerkin formulations [6,7]. It has been shown that the method can yield not only accurate lower mode frequencies but also accurate higher mode frequencies, and thus is the most efficient method among many others for analyzing the challenge problems of beams and plates vibrating at high modes [7–12].

Since the DSC was proposed by Wei in 1999, the DSC has been successfully used to analyze the static, vibration and buckling of beams, plates and shells. To name a few, the DSC is used to analyze various beam problems [13–18], plate problems [6,7,19–27], shell problems [28–31], wave propagation in rod [32] and non-linear buckling of drill string [33].

It is an easy task to implement boundary conditions by weak form methods, such as the finite element method and DSC-Ritz element method [26]. However, implementation of the boundary conditions by strong form methods, such as the DSC, is not an easy but an important task. The well-known methods are the method of symmetric extension used to apply the fixed boundary conditions and the method of anti-symmetric extension used to apply the simply supported boundary conditions [7]. However, these two methods cannot be used to apply the free boundary conditions. Therefore, most of early applications of the DSC considered only the boundary conditions of the clamped, simply supported and their combinations. The iteratively matched boundary (IMB) method, which applies boundary conditions at a free end repeatedly, can yield accurate lower mode frequencies for beams with free ends [14]. The Taylor series expansion method [11], can yield accurate lower mode as well as higher mode frequencies for beams with free ends [16–18]. However, the solution accuracy losses somehow when the method is used for rectangular plates with free corners. The matched interface and boundary (MIB) method is generalized for the

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free vibration analysis of rectangular plates [22]. A set of new MIB schemes are proposed and the boundary conditions are rigorously enforced. However, the lower mode frequencies seem also not very accurate. Besides its performance on higher mode frequency has not been demonstrated.

The objectives of this investigation are two folds. One is to provide a general and accurate way to apply any kind of boundary conditions by using the DSC. The other is to propose a simple way to overcome the difficulty existing in early research by the DSC. Free vibration of beams and rectangular plates with free boundaries are solved by the proposed DSC. Both lower mode and higher mode frequencies are investigated. For verifications, results are compared with exact solutions, finite element data and results obtained by the differential quadrature method (DQM) with much higher number of grid points. Based on the results reported herein, some conclusions are drawn.

2. Theories of slender beams and thin rectangular plates

2.1. Free vibration of slender beams

For an Euler–Bernoulli beam with length L, width b, height h and cross-sectional area A, the governing equation for free vibration analysis is

$$EI\frac{d^4w(x)}{dx^4} = \rho A\omega^2 w(x) \tag{1}$$

where x is the Cartesian coordinate in the middle plane of the beam, $w(x)$ is the deflection, $I = bh^3/12$ is the flexural rigidity, E is the modulus of elasticity, ρ is the mass density, and ω is the circular frequency.

The equations of three classical boundary conditions considered are

(1) Clamped end (C)

$$w = \frac{dw}{dx} = 0 \quad (x = 0 \text{ or } x = L) \tag{2}$$

(2) Simply supported end (S)

$$w = EI\frac{d^2w}{dx^2} = 0 \quad (x = 0 \text{ or } x = L) \tag{3}$$

(3) Free end (F)

$$EI\frac{d^2w}{dx^2} = EI\frac{d^3w}{dx^3} = 0 \quad (x = 0 \text{ or } x = L) \tag{4}$$

2.2. Free vibration of isotropic thin rectangular plates

For an isotropic thin rectangular plate with length a, width b and thickness h, the governing equation for free vibration analysis is given by

$$\frac{\partial^4 w(x,y)}{\partial x^4} + 2\frac{\partial^4 w(x,y)}{\partial x^2\partial y^2} + \frac{\partial^4 w(x,y)}{\partial y^4} = \frac{\rho h\omega^2 w(x,y)}{D} \tag{5}$$

where x and y are the Cartesian coordinates in the middle plane of the plate, $w(x, y)$ is the deflection, ρ is the mass density, $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity, E and ν are the modulus of elasticity and Poisson's ratio, and ω is the circular frequency, respectively.

The equations of three classical boundary conditions considered are

(1) Clamped edge (C)

$$w = \frac{\partial w}{\partial x} = 0 \quad (x = 0 \text{ or } x = a), \tag{6}$$

$$w = \frac{\partial w}{\partial y} = 0 \quad (y = 0 \text{ or } y = b), \tag{7}$$

(2) Simply supported edge (S)

$$w = D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = 0 \quad (x = 0 \text{ or } x = a) \tag{8}$$

$$w = D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) = 0 \quad (y = 0 \text{ or } y = b) \tag{9}$$

(3) Free edge (F)

$$D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x\partial y^2}\right] = 0 \quad (x = 0 \text{ or } x = a) \tag{10}$$

$$D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) = D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial y\partial x^2}\right] = 0 \quad (y = 0 \text{ or } y = b) \tag{11}$$

At a free corner point, one more condition should be satisfied, namely,

$$R = 2D(1-\nu)\frac{\partial^2 w}{\partial x\partial y} = 0 \tag{12}$$

3. DSC algorithm and solution procedures

3.1. DSC algorithm

For completeness consideration, the discrete singular convolution (DSC) is briefly described herein. In the DSC, the function $w(x)$ and its n^{th} order derivative with respect to x are approximated via a discretized convolution as [4,7],

$$w^{(n)}(x) \approx \sum_{k=-M}^{M} \delta_{\sigma,\Delta}^{(n)}(x - x_k)w(x_k) \quad (n = 0, 1, 2, \cdots) \tag{13}$$

where x_k are the uniformly distributed grid points, superscripts (n) denote the nth order derivative with respect to x, $2M + 1$ is called the computational bandwidth, $\delta_{\sigma,\Delta}(x - x_k)$ is a collective symbol for the delta kernels of Dirichlet type and its nth order derivative with respect to x is given by

$$\delta_{\sigma,\Delta}^{(n)}(x - x_k) = \left(\frac{d}{dx}\right)^n \delta_{\sigma,\Delta}(x - x_k) \tag{14}$$

There are several delta kernels of Dirichlet type available. The most widely used two kernels are the regularized Shannon kernel (DSC-RSK) and the non-regularized Lagrange's delta sequence kernel (DSC-LK). The regularized Shannon kernel is defined as [4]

$$\delta_{\sigma,\Delta}(x - x_k) = \frac{\sin[(\pi/\Delta)(x - x_k)]}{(\pi/\Delta)(x - x_k)}\exp\left[-\frac{(x - x_k)^2}{2\sigma^2}\right] \tag{15}$$

where σ is a controllable parameter, $\Delta = x_k - x_{k-1}$ is the spacing between two grid points.

The non-regularized Lagrange's delta sequence kernel is given by

$$\delta_{\sigma,\Delta}(x - x_k) = \begin{cases} L_{M,k}(x - x_k) & \text{for } -\beta \le x \le \beta \\ 0 & \text{otherwise} \end{cases} \quad \text{for } M = 1, 2, \ldots \tag{16}$$

where $\beta \le L_c$ (L_c equals to the length of beam L, plate length a or plate width b), and $L_{M,k}(x)$ is the Lagrange interpolation defined by [7]

$$L_{M,k}(x) = \prod_{i=k-M, i\ne k}^{k+M} \frac{x - x_i}{x_k - x_i} \quad (M \ge 1) \tag{17}$$