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Functionally graded rod with small concentration of inclusions: Homogenization and optimization

Igor V. Andrianov[a], Jan Awrejcewicz[b,c,*], Alexander A. Diskovsky[d]

[a] Institut für Allgemeine Mechanik, RWTH Aachen University, Templergraben 64, D-52056 Aachen, Germany
[b] Department of Automation, Biomechanics and Mechatronics, Lodz University of Technology, 1/15 Stefanowskiego St., PL-90924 Lodz, Poland
[c] Department of Vehicles, Warsaw University of Technology, 84 Narbutta Str., PL-02524 Warsaw, Poland
[d] National Metallurgical Academy of Ukraine, Department of Higher Mathematics, Gagarina 23, UA-49005 Dnepr, Ukraine

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ABSTRACT

This work is devoted to strain analysis and optimal design of a Functionally Graded (FG) rods and beams with small inclusions. The homogenization procedure plays a key role in our investigations. The method is illustrated using an example of the rod longitudinal deformation and bending of a beam. We consider the cases of FG inclusion sizes and FG steps between inclusions separately. Particular problems of optimal design are discussed in some details. The mathematical model of the bending beam, which adapts to the external load action, is proposed and an illustrative example of the adaptation process is given.

1. Introduction

The mechanical response of materials with spatial gradients in composition and structure is of considerable interest in numerous and diverse disciplines, such as tribology [1], geology [2,3] optoelectronics, biomechanics [4,5], fracture mechanics [6], and nanotechnology [7,8]. A fundamental approach allowing for deduction of the macro-scale laws and the constitutive relation by proper homogenization over the micro-scale is known as the homogenization method [9–17]. This method is also successfully used for modeling and simulating mechanical behavior of the FG Materials (FGM) [18,19] and the Functionally Graded Structures (FGS). Typically the term FGS is associated with the constructions made/fabricated from FGM. However, in this paper, the term FGS is understood in a broader manner, since the heterogeneous constructions with a controlled heterogeneity parameter are also taken into account (for instance, the reinforced plates and shells with nonuniformly distributed ribs of different stiffness; goffer-type constructions consisting of different amplitudes of goffer shapes and their half-wave length, etc. [20–25]). FGMs are composites consisting of two different materials with a gradient composition. In the case of application of the homogenization method, the coefficients of periodic composites state equations are usually [9–13] approximated by the first terms of their Fourier series (Fig. 1a). In a similar way [20–25], the coefficients of FGSs state equations with FG inclusion sizes (Fig. 1b) and FG step between the inclusions (Fig. 1c) can be approximated.

However, the truncated Fourier series (even for a small number of terms) relatively well approximate the coefficients of the constitutive equations for large concentration of inclusions (fibres, ribs, etc.), when the distance between inclusions is of the same order as their typical sizes. However, for small concentration, when the distance between inclusions is essentially larger than their size, the constitutive equation coefficients are approximated by impulse periodic function (see, for instance, Fig. 1d). In this case, a usual homogenization procedure may be accompanied by some problems to be directly applied.

Therefore, for a small concentration of inclusions, it is recommended to use the further presented variant of homogenization method, where small sizes of the inclusions with respect to the distance between them are utilized to employ the asymptotic procedure. Modifications of this approach for FGS with small inclusion concentrations are also proposed.

The applied method is illustrated using a relatively simple problem, i.e., we consider a rod with a longitudinal strain. In our investigations, the rod diameter is taken commensurable with inclusions sizes.

2. FG inclusion sizes

FG properties can be achieved, for instance, by applying different inclusion sizes. Let us analyze an influence of different sizes of inclusions on the longitudinal rod stiffness, keeping the distance between inclusions (Fig. 2) constant. We define changes of the inclusion

* Corresponding author.
E-mail addresses: igor_andrianov@hotmail.com (I.V. Andrianov), awrejcew@p.lodz.pl (J. Awrejcewicz), alex_diskovskiy@ukr.net (A.A. Diskovsky).

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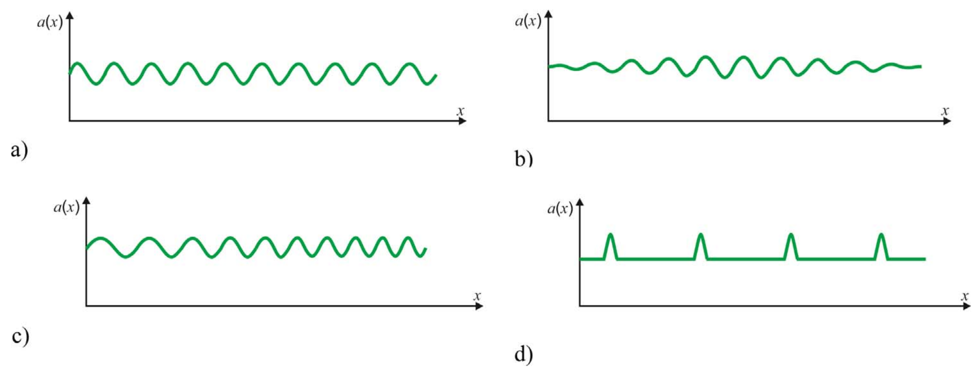

Fig. 1. Schematic view of the constitutive equation coefficient $a(x)$for a composite: a) periodic structure; b) FG inclusion sizes; c) FG steps between inclusions; d) small inclusion concentration.

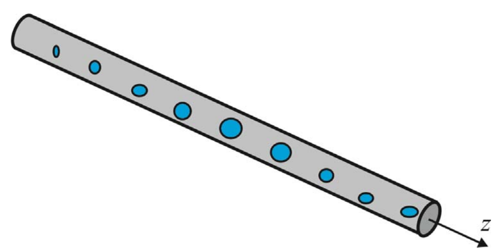

Fig. 2. Schematic view of the rod with FG sizes of inclusions.

dimensions by a function $V = V(x)$.

In what follows, we consider a deformation of the FGM rod subjected to the spatially distributed load $P(z)$ and the inclusions (Fig. 2) by being equivalent to concentrated elastic elements (Fig. 3). Observe that for composites with regular structure, the analogous models of two-component rod are applied (see references [26–28]).

Obviously, a number of n is large, and hence the distance $l = z_i - z_{i-1}$ between them is much less than the rod length L, $l \ll L$. Therefore, in order to investigate the longitudinal deformation of the two-component rod (Fig. 3), one may apply the following variant of the homogenization procedure.

Equilibrium equation of the rod part between the concentrated elastic elements has the following form

$$\frac{d^2u}{dx^2} = p, \tag{1}$$

where: $x = z/l$; $u = v/l$; v is the longitudinal displacement; $p = \frac{P(lx)}{lk_0}$; $k_0 = E_0F$; E_0 is Young's modulus of the rod material; F is the cross section area.

Since the approximate inclusions composed of the elastic elements can be treated as discrete elastic cross sections, the associated compatibility conditions regarding the i-th inclusion follow

$$(u)^+ = (u)^-; \quad \left(\frac{du}{dx}\right)^+ - \left(\frac{du}{dx}\right)^- = ku, \tag{2}$$

where $(\ldots)^- = \lim_{x=i-0}(\ldots)$; $(\ldots)^+ = \lim_{x=i+0}(\ldots)$; $k = \frac{lk_1(x)}{k_0}$; k_1 is the stiffness of the discrete elastic inclusions.

3. Homogenization procedure for longitudinal deformation

Owing to the homogenization approach, let us introduce the "fast" variable ξ

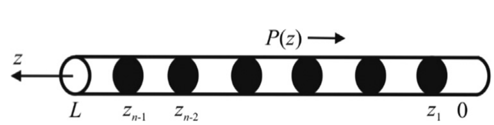

Fig. 3. Schematic view of the two-component rod with concentrated elastic elements.

$$\xi = x/\varepsilon, \tag{3}$$

where $\varepsilon = 1/n \ll 1$.

We treat the variables x and ξ as independent ones, and the differential operator used in (1), (2) has the following form

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \varepsilon^{-1}\frac{\partial}{\partial \xi}. \tag{4}$$

Displacement u can be presented in the following form

$$u = u_0(x) + \varepsilon^2 u_1(x, \xi) + \varepsilon^3 u_2(x, \xi) + \ldots, \tag{5}$$

where u_s ($s = 1,2,\ldots$) is a periodic function with respect to ξ with period n.

Substituting Ansatzes (4), (5) into Eq. (1) and compatibility condition (2) and carrying out the splitting with respect to ε, the following homogenized equation describing the longitudinal displacement of the two-component rod is obtained

$$\frac{d^2u_0}{dx^2} + k(x)u_0 = p. \tag{6}$$

Micromechanical effects are described by the functions u_s ($s = 1,2, \ldots$). For the function u_1 on the period $\xi \in (0, n)$one obtains:

$$\frac{\partial u_1}{\partial \xi} = k(x)u_0\left(\xi - \frac{n}{2}\right). \tag{7}$$

Next, the function u_1 is periodically extended along the whole rod length.

4. Inverse problem

The main advantage of the proposed approach is that it allows to efficiently solve the problems of optimization, i.e., problems devoted to determination of optimal characteristics of the internal material structure protecting the given structure properties. In the studied case of the FG amplitudes, the target characteristic is the function $V = V(x)$ governing a rule of the inclusion sizes change. As an example we consider the problem of determination of the function $V(x)$ that provides the largest longitudinal stiffness of the rod under a given load.

It is convenient to rather take the function $k(x)$ as the control function instead of the function $V(x)$.

Without loss of generality, let us take the boundary conditions in the following form

$$u|_{x=0} = 0, \quad \frac{du}{dx}|_{x=n} = 0. \tag{8}$$

In order to measure the rod stiffness properties, we take energy of the elastic deformations and use zero-order approximation of the displacement (5). Then, we define a minimum of the following functional