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Analytical modeling of constricted channel flow

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ABSTRACT

Analytical flow models are frequently applied when describing constricted channel flow at low and moderate Reynolds numbers. A common assumption underlying such flow models is two-dimensional or axi-symmetrical flow. In this work, two analytical model approaches are formulated in order to overcome this assumption in the case of naturally occurring channel flows for which the assumption might be critiqued. Advantages and flaws of both model approaches are discussed and their outcome is compared with experimental data.

1. Introduction

Many applications rely on simplified laminar models to obtain an estimation of quasi-steady flow through constricted channels at a low computational cost. For low or moderate Reynolds numbers, viscous flow effects, which are known to depend on the cross-section shape [1,2], potentially affect the flow field. Nevertheless, common simplified models often rely on the assumption of two-dimensional or axi-symmetrical flow so that the cross-section shape is neglected. Imaging studies of naturally occurring constricted channel flows, such as physiological flow through blood vessels or airways, revealed a large variation of channel's cross-section shapes so that the assumption of two-dimensional (2D) or axi-symmetrical flow can be questioned for these applications [3]. In the following, two analytical flow models are considered which account for the cross-section shape of the constricted channel portion so that both result in 'quasi-three-dimensional' (quasi-3D) flow models. The first model (boundary layer model) makes the assumption of developing boundary layers whereas the second model (viscous model) relies on the asymptotic case of fully developed boundary layers.

2. Constricted channel flow

Pressure driven quasi-steady flow through a constricted channel (Fig. 1) is considered. A uniform circular channel (area A_0) envelops a constricted portion with minimum constriction (area A_c, length L_c, hydraulic diameter $D = 4A_c/\mathcal{P}$ with $\mathcal{P}$ the wetted perimeter) for which viscous effects can not be neglected. All corners are rounded (radius r_a). Flow is then generated by imposing upstream pressure P_0 so that the total driving pressure difference yields $\Delta P = P_0 - P_d$ with downstream pressure $P_d = 0$. Jet formation occurs near the downstream end of the constricted region (x_s) where the flow separates from the channel wall. The pressure distribution $P(x, t)$ along the constricted channel portion ($x_0 \leq x_2$) is sought for a known value of upstream pressure P_0.

Experimental data are obtained as described in [4]. Concretely, upstream pressure P_0, pressure P_1 at the middle of the constriction ($x/L_c = 0.5$) and volume flow velocity Φ are measured. In addition, spatial velocity profiles u are measured along (longitudinal – $u(x)$) and perpendicular (spanwise – $u(y)$) to the main flow direction. Mean values are considered which are derived on 5 s of steady signal for the measured pressure signal $P(t)$ and volume flow rate $\Phi(t)$ and on 40 s for velocity $u(t)$.

3. Quasi-3D analytical laminar flow modeling

Low or moderate Reynolds number quasi-steady airflow (kinematic viscosity $\nu = 1.5 \times 10^{-5}\,\mathrm{m^2/s}$ and density $\rho = 1.2\,\mathrm{kg/m^3}$) is considered so that the flow within the constriction is assumed laminar and incompressible. The no-slip boundary condition is applied on the rigid channel walls. Volume flow velocity Φ is conserved so that $d\Phi/dx = 0$. Two cases are considered based on the ratio of constriction length to the entrance length of the constriction required to obtain fully developed viscous flow [1,2]. In the first case (Section 3.1), L_c is short or comparable to the entrance length of the constricted portion so that viscous boundary layers develop within the constriction. Downstream from the constriction pressure recovery due to flow mixing is accounted for so that the expanding jet

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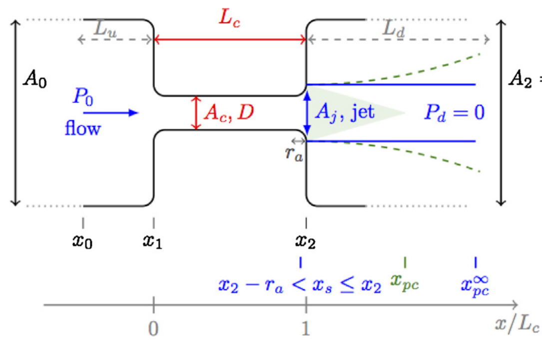

Fig. 1. Illustration of pressure driven flow through a uniform circular channel (area A_0) enveloping a constricted portion (area A_c, hydraulic diameter D and length L_c). Sharp edges are rounded (radius r_a). Main streamwise direction x, pressure upstream from the constriction P_0, pressure downstream from the constriction P_d, flow separation position x_s, upstream unconstricted channel portion length (L_u) and downstream unconstricted channel portion length (L_d) are indicated. A non-expanding stable straight jet (full lines) with infinite potential core extent x_{pc}^{∞} and a developing jet (dashed curved lines) with finite potential core extent x_{pc} (shaded area) are depicted.

has a finite potential core x_{pc} (Fig. 1). In the second case (Section 3.2), L_c is long compared to the entrance length of the constricted portion so that fully developed boundary layers are accounted for. Flow separation is discussed in Section 3.3.

3.1. Boundary layer model

A simple boundary layer flow model is proposed accounting for a developing boundary layer enveloping the core flow region. Pressure recovery due to flow mixing of the jet issued from the constriction with the surrounding fluid downstream from the constriction is accounted for using conservation of mass and momentum over the mixing region:

$$u_j A_j = u_2 A_2, \tag{1}$$

$$\rho u_2^2 A_2 = P_j A_2 + \rho A_j u_j^2, \tag{2}$$

where subindex j and 2 indicate respectively the jet region (cross-sectional area A_j, velocity u_j and pressure P_j) and the region downstream from the mixing zone (cross-sectional area $A_2 = A_0$, velocity u_2 and pressure $P_2 = P_d = 0$). The jet cross-sectional area A_j is given as

$$\frac{A_j}{A_c} = \left(1 - \frac{2\delta_1}{D}\right)^2 \quad \text{or} \quad \frac{A_j}{A_c} \approx 1 - \frac{4\delta_1}{D} \quad \text{since} \quad \frac{2\delta_1}{D} < 1, \tag{3}$$

with δ_1 the displacement thickness of the boundary layer approximated as the value for a flat plate of length L_c associated with a Blasius velocity profile [2]:

$$\delta_1 \approx 1.7\sqrt{\frac{L_c D}{Re_{ref}}}, \tag{4}$$

where reference Reynolds number $Re_{ref} = \frac{D u_{ref}}{\nu}$ is defined using hydraulic diameter D and reference velocity $u_{ref} = \sqrt{\frac{2P_0}{\rho}}$. An estimation of the pressure within the jet P_j yields

$$\frac{P_j}{P_0} = \frac{-2\frac{A_j}{A_2}\left(1 - \frac{A_j}{A_2}\right)}{1 - \frac{2A_j}{A_2}\left(1 - \frac{A_j}{A_2}\right)} \tag{5}$$

and the pressure drop $\Delta P_c = P_c - P_j$ becomes

$$\frac{\Delta P_c}{P_0} = \frac{P_0 - P_j}{P_0}\left(1 - \frac{A_j^2}{A_c^2}\right). \tag{6}$$

The pressure within the constriction is then estimated as $P(0 \leq x \leq L_c) \approx P_j + \frac{x\Delta P_c}{L_c}$ so that the pressure at the center ($x/L_c = 0.5$) of the constriction is approximated as $P_1 \approx P_j + \frac{\Delta P_c}{2}$.

The centerline velocity u within the constriction ($0 \leq x \leq L_c$), *i.e.* in the core flow region outside the boundary layer, is estimated by approximating the area $A(x)$ following (3) and (4) as

$$\frac{A(x)}{A_c} = \left(1 - \frac{2\delta_1(x)}{D}\right)^2 \quad \text{with} \quad \delta_1(x) \approx 1.7\sqrt{\frac{xD}{Re_{ref}}} \tag{7}$$

so that

$$u(x) \approx \frac{\Phi}{A(x)} \tag{8}$$

with volume flow velocity Φ estimated as

$$\Phi \approx u_{ref} \cdot \bar{A(x)}, \tag{9}$$

where $\bar{A(x)}$ indicates the mean value of $A(x)$ within the constriction using (7). Consequently, flow quantities within the constriction are estimated using a single input parameter (upstream pressure P_0) while accounting for the cross-section shape by its hydraulic diameter D. Note that downstream from the constriction within the potential core of the jet ($L_c < x \leq x_{pc}$) both the velocity and pressure can be considered constant so that $u(x) \approx \frac{\Phi}{A_j}$ and $P(x) \approx P_j$.

3.2. Viscous model

The streamwise momentum equation of the governing Navier–Stokes equation for driving pressure dP/dx is approximated using volume flow velocity conservation $d\Phi/dx = 0$ as [4,5]:

$$-\frac{\Phi^2}{A^3}\frac{dA}{dx} + \frac{1}{\rho}\frac{dP}{dx} = \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{10}$$

with spanwise direction y, transverse direction z and velocity $u(x, y, z)$. The flow model expressed in (10) accounts for viscosity (right hand term) as well as flow inertia (first source term at the left hand side) and depends therefore on the area as well as on the shape of the cross-section. It is seen that for a uniform channel, so that $dA/dx = 0$ holds, (10) reduces to purely viscous flow [5,2]. The same way, it is seen that (10) reduces to Euler's equation describing Bernoulli flow when viscosity is neglected, *i.e.* $\nu = 0$ as for an ideal inviscid flow [2].

The pressure distribution $P(x, t)$ as a function of streamwise position x and time t up to flow separation ($x_0 \leq x \leq x_s$) is then given by integration of (10) [5,4] and results in a quadratic equation of volume flow velocity Φ:

$$\begin{aligned} P(x,t) &= P_0 + \frac{1}{2}\rho\Phi^2\left(\frac{1}{A^2(x_0)} - \frac{1}{A^2(x,t)}\right) \\ &\quad + \mu\Phi\int_{x_0}^{x}\frac{dx}{\beta(x,t)}, \qquad \text{if} \;\; x_0 \leq x < x_s, \end{aligned} \tag{11}$$

with dynamic viscosity of the fluid $\mu = \rho\nu$ and β expressing the viscous contribution to the pressure drop so that it depends on the cross-section shape. The assumption of a stable non-expanding straight jet with infinite potential core extent x_{pc}^{∞}, *i.e.* non-viscous flow, results in $P(x, t) = P_d$ downstream from flow separation ($x \geq x_s$). From (11) is seen that the model adds a viscous correction (last righthand term) to the steady Bernoulli equation [1,2] which relies on the asymptotic expression for fully developed viscous flow.