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Letter

Buckling of thin gel strip under swelling

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HIGHLIGHTS

- An analytical model is established for the swelling-induced buckling of a thin gel strip.
- The closed-form solutions for the amplitude and wavelength are obtained.
- The results provide design guidelines for the buckled configuration quantitatively.

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ABSTRACT

The buckling of thin gel film has attracted much attention due to its applications in the design of three-dimensional structure from two-dimensional template. We have established an analytical model to study the swelling-induced buckling of a thin gel strip with one edge clamped and the others free. The closed-form solutions for the amplitude and wavelength of the buckled shape are obtained by energy minimization of the total potential energy. The analytical results agree well with finite element analysis based on the inhomogeneous gel theory without any parameter fitting. The model provides a route to study complex postbuckling behaviors of thin gel films and guidelines to design the buckled configuration quantitatively by controlling the swelling ratio.

The swelling of planar thin gel films could induce off-plane deformation due to buckling and has emerged as an efficient route for the design of three-dimensional structure from two-dimensional template [1–3]. Depending on the geometries and constrains, various buckling patterns have been observed. For example, for a thin annular plate with its inner edge clamped and outer edge free, the gel swelling can induce a periodic buckling pattern in a sinusoidal form along the outer free edge [4–6]. For a thin rectangular strip with one edge clamped and the others free, similar buckling pattern with the periodic sinusoidal out-of-plane deformation along the free edge can be observed [4,5]. The recent work by introducing inhomogeneous swelling in the gel film has realized much more complex three-dimensional shapes such as ziggurat, sheave, drum, and smiling face [3,7].

Although the theoretical framework for the polymeric gel [8] is successful to explain the swelling-induced buckling of thin film, closed-form solutions only exist for the threshold of buckling, i.e., initial buckling, to obtain the critical buckling load and buckling shape. Most of the existing results on postbuckling are from experiments and numerical simulations [4–6,9,10] even for simple geometries such as rectangular strip and annular plate. Since a gel can be treated as a hyperelastic solid [8] when it is in inhomogeneous equilibrium, closed-form solutions for the postbuckling of a swelling gel are possible to be obtained based on the well-established nonlinear buckling mechanics theory for an elastic thin film with reasonably close agreement.

The objective of this paper is to establish an analytical model, validated by finite element analysis (FEA) based on the inhomogeneous gel theory, for the swelling-induced buckling of thin gel films. A rectangular thin strip with one edge constrained and the others free as shown in Fig. 1(a) is used to illustrate the approach.

The initial buckling analysis was given earlier by Mora and Boudaoud [4] and the main results are summarized here. The thin gel strip with length l, width b, and thickness h as shown in Fig. 1(a), where $l \gg b$ and $l \gg h$, can be modeled as a von Karman plate. The Cartesian coordinates x_1, x_2, and x_3 are along the length, width, and thickness directions, respectively. The bottom edge ($x_2 = 0$) is fixed, and the top edge ($x_2 = h$) is free. Let γ denote the swelling ratio. As γ increases to exceed a critical value, the gel strip buckles

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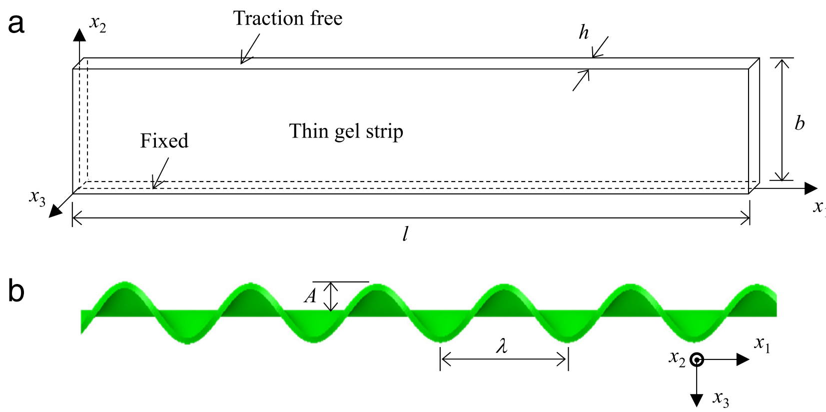

Fig. 1. (a) Schematic diagram of a thin gel strip with one edge clamped and the others free. (b) Schematic diagram of the swelling-induced buckled thin gel strip.

to form a three-dimensional deformation pattern, as shown in Fig. 1(b), defined by the in-plane displacements $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$, and the out-of-plane displacement $w(x_1, x_2)$.

The out-of-plane displacement $w(x_1, x_2)$ can be expressed as $w = f(x_2)\sin(kx_1)$, which satisfies the equilibrium equation

$$\frac{\partial^4 w}{\partial x_1^4} + 2\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 w}{\partial x_2^4} = -\frac{N_{11}}{D}\frac{\partial^2 w}{\partial x_1^2}, \tag{1}$$

where N_{11} is the membrane force in x_1 direction, k is the wavenumber, and $D = Eh^3/\left[12\left(1-\nu^2\right)\right]$ is the bending rigidity with E as Young's modulus and ν as Poisson's ratio. Equation (1) then yields the solution of $f(x_2)$ as

$$f(x_2) = C_1 \mathrm{e}^{-\alpha x_2} + C_2 \mathrm{e}^{\alpha x_2} + C_3\cos(\beta x_2) + C_4\sin(\beta x_2), \tag{2}$$

where $\alpha = \sqrt{\sqrt{\frac{N_{11}}{D}k^2} + k^2}$, $\beta = \sqrt{\sqrt{\frac{N_{11}}{D}k^2} - k^2}$, and $C_i\,(i = 1, 2, 3, 4)$ are constants to be determined by the boundary conditions on the clamped ($x_2 = 0$) and free edges ($x_2 = b$). At $x_2 = 0$, w satisfies $w = 0$ and $\partial w/\partial x_2 = 0$, which give $C_1 = (\beta C_4 - \alpha C_3)/(2\alpha)$ and $C_2 = -(\alpha C_3 + \beta C_4)/(2\alpha)$. At $x_2 = b$, w satisfies $\partial^2 w/\partial x_2^2 + \nu\partial^2 w/\partial x_1^2 = 0$ and $\partial^3 w/\partial x_2^3 + (2-\nu)\,\partial^3 w/(\partial x_1^2 \partial x_2) = 0$, which give a system of 2 linear equations with the two unknowns C_3 and C_4 as

$$\begin{bmatrix} \begin{array}{l}\left(\beta^2+\nu k^2\right)\cos(\beta b)\\ \quad+\left(\alpha^2-\nu k^2\right)\cosh(\alpha b)\end{array} & \begin{array}{l}\left(\beta^2+\nu k^2\right)\sin(\beta b)\\ \quad+\dfrac{\beta}{\alpha}\left(\alpha^2-\nu k^2\right)\sinh(\alpha b)\end{array} \\ \begin{array}{l}\beta\left(-\beta^2+(2-\nu)k^2\right)\sin(\beta b)\\ +\alpha\left(\alpha^2-(2-\nu)k^2\right)\sinh(\alpha b)\end{array} & \begin{array}{l}\beta\left(\beta^2+(2-\nu)k^2\right)\cos(\beta b)\\ +\beta\left(\alpha^2-(2-\nu)k^2\right)\cosh(\alpha b)\end{array} \end{bmatrix} \times \begin{Bmatrix} C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{3}$$

A nonzero solution exists only if the determinant is zero, which yields the following equation:

$$\begin{aligned} &2\alpha\beta\left(\alpha^2k^2 - 2\nu k^4 + \nu^2k^4 - \beta^2k^2 + \alpha^2\beta^2\right)\cos(\beta b)\cosh(\alpha b)\\ &+\begin{pmatrix} 4\alpha^2\beta^2k^2 - 2\nu\alpha^2\beta^2k^2 - \nu\alpha^4k^2 + 2\nu\alpha^2k^4\\ -\nu^2\alpha^2k^4 - \nu\beta^4k^2 - 2\nu\beta^2k^4 + \nu^2\beta^2k^4\\ -\alpha^4\beta^2 + \alpha^2\beta^4 \end{pmatrix}\\ &\times\sin(\beta b)\sinh(\alpha b)\\ &-\alpha\beta\left(2\alpha^2k^2 - 2\beta^2k^2 - 4\nu k^4 + 2\nu^2k^4 - \alpha^4 - \beta^4\right) = 0. \end{aligned} \tag{4}$$

By introducing the non-dimensional wave number $\bar{k} = kb$ and normalized membrane compressive force $\bar{N} = N_{11}b^2/D$, α and β become $\alpha = \sqrt{\bar{k}\left(\sqrt{\bar{N}}+\bar{k}\right)}/b$ and $\beta = \sqrt{\bar{k}\left(\sqrt{\bar{N}}-\bar{k}\right)}/b$. Equation (4) then becomes a nonlinear equation $\bar{g}\left(\bar{k}, \bar{N}, \nu\right) = 0$. For a given Poisson's ratio ν, the nonlinear equation is in terms of $\bar{N}$ and $\bar{k}$. For a given $\bar{k}$, there are many solutions of $\bar{N}$ which satisfy the nonlinear equation. We choose the smallest $\bar{N}$ as the solution and plot the curve of $\bar{N}$ versus $\bar{k}$. The minimum point on the curve then gives the critical $\bar{N}_c$ and the critical $\bar{k}_c$. Once $\bar{N}_c$ and $\bar{k}_c$ are found, the $C_i\,(i = 1, 2, 3, 4)$, α and β can also be obtained. For example, for the incompressible gel ($\nu = 0.5$), $\bar{N}_c = 10.399$ and $\bar{k}_c = 1.930$, which give $C_1 = -0.9593$, $C_2 = -0.04067$, $C_3 = 1$, $C_4 = -1.833$, $\alpha = 3.154/b$, and $\beta = 1.581/b$. The buckling process can be viewed as a free swelling γ followed by a compression $\varepsilon = (\gamma - 1)/\gamma$ in the length direction. The corresponding critical wavelength λ_c and critical swelling ratio γ_c are then given by

$$\lambda_c = 3.256b, \quad \gamma_c = \frac{1}{1 - 1.155h^2/b^2}. \tag{5}$$

The out-of-plane displacement can be written as

$$\begin{aligned} w = \zeta\,&\left[C_1\mathrm{e}^{-\alpha x_2} + C_2\mathrm{e}^{\alpha x_2} + C_3\cos(\beta x_2)\right.\\ &\left.+ C_4\sin(\beta x_2)\right]\sin(kx_1), \end{aligned} \tag{6}$$

where the parameter ζ is to be determined in the postbuckling analysis described below.

The postbuckling behavior is studied by energy minimization of the total potential energy of the strip. The membrane strains are related to the in-plane displacements $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$, and out-of-plane displacement $w(x_1, x_2)$ by

$$\begin{aligned} \varepsilon_{11} &= u_{1,1} + \frac{1}{2}w_{,1}^2 - \varepsilon,\\ \varepsilon_{22} &= u_{2,2} + \frac{1}{2}w_{,2}^2 + \nu\varepsilon,\\ \varepsilon_{12} &= \frac{1}{2}\left(u_{1,2} + u_{2,1}\right) + \frac{1}{2}w_{,1}w_{,2}. \end{aligned} \tag{7}$$

Hooke's law gives the membrane forces in the thin film as

$$\begin{aligned} N_{11} &= \bar{E}h\left(\varepsilon_{11} + \nu\varepsilon_{22}\right),\\ N_{22} &= \bar{E}h\left(\varepsilon_{22} + \nu\varepsilon_{11}\right),\\ N_{12} &= \bar{E}h\,(1-\nu)\,\varepsilon_{12}, \end{aligned} \tag{8}$$

where $\bar{E} = E/\left(1-\nu^2\right)$ is the plane strain modulus of film. The equilibrium equations are

$$\begin{aligned} \frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} &= 0,\\ \frac{\partial N_{21}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} &= 0. \end{aligned} \tag{9}$$