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Conservation laws and non-invariant solutions of anisotropic wave equations with a source

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HIGHLIGHTS

- Nonlinearly self-adjoint anisotropic wave equations are classified.
- Conservation laws for wave equations with a source are found.
- Non-invariant exact solutions are computed.
- Solutions are constructed by the method of conservation laws.

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ABSTRACT

Linear and nonlinear waves in anisotropic media are used in various fields, e.g. in biomechanics, biomedical acoustics, etc. The present paper is devoted to discussion of nonlinear anisotropic wave equations with a source from point of view of their conservation laws and exact solutions associated with conservation laws. Nonlinearly self-adjoint wave equations with special source terms are singled out.

The conservation laws associated with symmetries of the nonlinearly self-adjoint wave equations are computed and used for constructing exact solutions. The obtained solutions are different from group invariants solutions, in particular, from steady state and traveling wave solutions.

The paper is designed for the application oriented readers. Its main goal is to introduce readers, interested in solutions of mathematical models having real world applications, to the recent *method of conservation laws* for constructing exact solutions of partial differential equations using conservation laws.

1. Introduction

Anisotropic waves play a significant part while dealing with complex materials in physics, biomechanics, etc. For example, acoustic methods in medical diagnostics are based on properties of propagation of acoustic waves in anisotropic media. These properties and specific anisotropy of the tissue of human body allow to differentiate between normal state of a tissue and pathology (see [1,2] and the references therein). Semiconductor devices provide another vast research area for investigating anisotropic waves.

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We will discuss here conservation laws and related exact solutions of a nonlinear anisotropic wave equation with a source, namely the scalar anisotropic partial differential equations of the second order of the form

$$u_{tt} = (f(u)u_x)_x + (g(u)u_y)_y + (h(u)u_z)_z + q(u).$$

The function $q(u)$ is responsible for an external source.

The above equation is linear if the functions $f(u),\ g(u),\ h(u)$ are constants and the source term $q(u)$ is a linear function, i.e. $q(u) = ku + k_1$. In other words, the anisotropic wave equation is linear if the following four equations are satisfied simultaneously:

$$f'(u) = 0, \quad g'(u) = 0, \quad h'(u) = 0, \quad q''(u) = 0.$$

The present paper is devoted to nonlinear equations. Hence it is assumed that at least one of these equations is not satisfied.

For the sake of brevity, we restrict our consideration to the two-dimensional nonlinear anisotropic wave equations

$$u_{tt} = (f(u)u_x)_x + (g(u)u_y)_y + q(u). \tag{1}$$

Eq. (1) with arbitrary functions $f(u),\ g(u)$ and $q(u)$ remains invariant after adding to each independent variables t, x, y any constant parameter. In other words, Eq. (1) admits the three-dimensional Lie algebra spanned by the following generators of the translation groups in t, x, y:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y} \cdot \tag{2}$$

The operators (2) are called the infinitesimal symmetries of Eq. (1).

Application of classical methods of Lie group analysis (see, e.g. [3]) to linear combinations of the infinitesimal symmetries (2) of Eq. (1) allows to construct only simple invariant solutions, such as stationary solutions, traveling waves, one-dimensional or two-dimensional solutions.

In the present paper we find the particular forms of the source term $q(u)$ when Eq. (1) has a conservative form. For this purpose, we use the formal Lagrangian approach to variational formulation of differential equations by extending the space of dependent variables u to the space (u, v) with auxiliary (*adjoint*) variables v [4,5]. The formal Lagrangian approach allows to extend Noether's conservation theorem to systems of differential equations not having a classical Lagrangian and to non-Noether symmetries [6].

The paper [7] is devoted to development of a combination of the generalized conservation theorem with the discrete variational principle and determination of discrete conservation laws. It is worth noting that the adjoint variables appear also in investigating integrating factors for higher-order ordinary differential equations [5,8].

The conserved vectors given by the new conservation theorem (Theorem 3.5 in [6]) contain the adjoint variables v. They can be eliminated if differential equations under consideration are *nonlinearly self-adjoint* [9]. Therefore, in Section 2 we investigate Eq. (1) for the nonlinear self-adjointness and single out the specific form of the source term $q(u)$ when Eq. (1) is nonlinearly self-adjoint.

Section 3 is devoted to computation of conservation laws for the nonlinearly self-adjoint equation (1) with a specific source term $q(u)$.

The method of conservation laws [9,10] for constructing exact solutions of partial differential equations using conservation laws is illustrated in Section 4.

The formal Lagrangian formalism has been applied currently to various mathematical models for constructing conservation laws and solutions, in particular to diffusion models [11,12] including a model of two-phase filtration of oil and gas in porous media [13], a system of dispersive evolution equations [14] as