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On periodic solutions to second-order Duffing type equations

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ABSTRACT

Sufficient and necessary conditions are found for the existence of a positive periodic solution to the Duffing type equation

$$u'' = p(t)u + q(t,u)u.$$

The results obtained are compared with facts well known for the autonomous Duffing equation

$$y'' + ay - by^3 = 0.$$

Uniqueness of solutions and possible generalizations are discussed, as well.

1. Introduction

The paper deals with the question on the existence and uniqueness of a positive solution to the periodic boundary value problem

$$u'' = p(t)u + q(t,u)u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega). \tag{1.1}$$

Here, $p \in L([0,\omega])$ and $q : [0,\omega] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Under a solution to problem (1.1), as usually, we understand a function $u : [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies given equation almost everywhere, and verifies periodic conditions.

A non-linear term of the form $q(t,x)x$ in (1.1) is chosen on the one hand with respect to convenient formulation of the results and on the other hand, since we are interested in problem (1.1) with super-linear non-linearities. Such kind of problems arises frequently in applications. For example, in mathematical models

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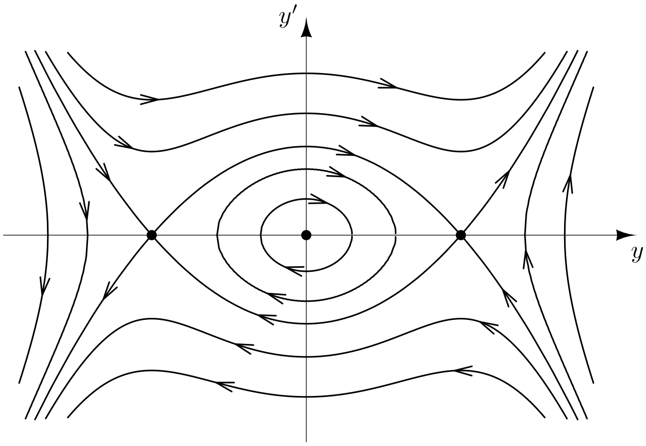

Fig. 1. Phase portrait of Eq. (1.4) with $a, b > 0$.

of various oscillators, one can find the following equation

$$y'' + \delta y' + ay - by^3 = \gamma \sin t, \tag{1.2}$$

where $a, b, \gamma \in \mathbb{R}$ and the damping constant satisfies $\delta \geq 0$. This equation is the central topic of the monograph [1] by Duffing published in 1918 and still bears his name today (see also [2]). Considering the equation of motion of the forced damped pendulum

$$y'' + \delta y' + \frac{g}{\ell} \sin y = \gamma \sin t. \tag{1.3}$$

Eq. (1.2) with $a, b > 0$ appears when approximating the non-linearity in (1.3) by Taylor's polynomial of the third order with the centre at 0. A survey of results dealing with the analysis of the pendulum equation is given in [3]. Eq. (1.2) can be also interpreted as the equation of motion of a forced oscillator with a spring whose restoring force is given as a third-order polynomial. The phase portrait of the free undamped Eq. (1.2) with $a, b > 0$, i. e., the equation

$$y'' + ay - by^3 = 0, \tag{1.4}$$

can be easily determined and it is illustrated in Fig. 1.

Definition 1.1. A solution u to problem (1.1) is referred as a sign-constant solution if there exists $i \in \{0, 1\}$ such that

$$(-1)^i u(t) \geq 0 \quad \text{for } t \in [0, \omega],$$

and a sign-changing solution otherwise.

Let us summarize some well-known facts concerning periodic solutions to Eq. (1.4) (see, e.g., [2,4]).

Proposition 1.2. *The following statements hold:*

(1) *For any $a \leq 0$ and $b > 0$, Eq.* (1.4) *has a unique equilibrium $y = 0$ and no other periodic solutions occur.*

(2) *For any $a, b > 0$, Eq.* (1.4) *has exactly three equilibria $y = 0$, $y = \sqrt{\frac{a}{b}}$, $y = -\sqrt{\frac{a}{b}}$ and no other non-trivial sign-constant periodic solutions occur.*