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A nonlinear eigenvalue optimization problem: Optimal potential functions

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ABSTRACT

In this paper we study the following optimal shape design problem: Given an open connected set $\Omega \subset \mathbb{R}^N$ and a positive number $A \in (0, |\Omega|)$, find a measurable subset $D \subset \Omega$ with $|D| = A$ such that the minimal eigenvalue of $-\mathrm{div}(\zeta(\lambda, x)\nabla u)+\alpha\chi_D u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, is as small as possible. This sort of nonlinear eigenvalue problems arises in the study of some quantum dots taking into account an electron effective mass. We establish the existence of a solution and we determine some qualitative aspects of the optimal configurations. For instance, we can get a nearly optimal set which is an approximation of the minimizer in ultra-high contrast regime. A numerical algorithm is proposed to obtain an approximate description of the optimizer.

1. Introduction

Let Ω be a bounded, connected, open set in $\mathbb{R}^N$ with Lipschitz boundary. Assume that A is a given positive number, $0 < A < |\Omega|$, where $|.|$ denotes the Lebesgue measure. Given a measurable set $D \subset \Omega$ with $|D| = A$, consider the following nonlinear eigenvalue problem

$$-\mathrm{div}(\zeta(\lambda, x)\nabla u) + \alpha\chi_D u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

In this paper, λ is the principal eigenvalue or the smallest positive eigenvalue of (1.1) and $u = u(x)$ is a corresponding eigenfunction.

We are interested in the cases that $\zeta(\lambda, x)$ is a nonlinear function of the parameter λ. Indeed, Eq. (1.1) can be regarded as a nonlinear elliptic eigenvalue problem because of the nonlinear dependence on the eigenparameter.

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Such nonlinear eigenvalue problems appear as the Hamiltonian equation governing some quantum dot nanostructures, where $\zeta(\lambda, x)$ corresponds to the effective mass of the carrier (electron or hole) and the surrounding matrix, $\alpha\chi_D$ is the potential function, λ is the ground state energy and u is the wave function [1–4]. A real physical phenomenon modeled by Eq. (1.1) is the heterostructures of different semiconductors where the electron effective mass depends on both the energy and position [3,4].

It is known that the ground state energy of (1.1) depends on the set D, the region with potential α, and we use the notation $\lambda(D)$ as we want to emphasize this dependence. To determine the system's potential which gives the minimum ground state energy, we consider the following optimization problem

$$\inf_{\substack{D\subset\Omega \\ |D|=A}} \lambda(D). \tag{1.2}$$

Let us recall here that nonlinear eigenvalue problems and optimization problems have many applications in engineering and applied sciences and these problems have been intensively attractive to mathematicians in the past decades [5]. However, it should be mentioned that the majority of the investigated nonlinear models are nonlinear in their differential operator part [6–9]. We note that Eq. (1.1) has nonlinear dependence on the parameter λ and such systems have been under less attention in this field of study [10–12].

Taking advantage of a variational characterization of the eigenvalues of a nonlinear eigenvalue problem [13,14], we derive in Section 2 the existence of an optimal ground state energy under certain conditions on the function ζ. Next we prove in Section 3 qualitative properties of the optimal shape $\hat{D}$. Namely, the optimal set contains a tubular neighborhood of the boundary $\partial\Omega$, and if Ω is simply connected and α is sufficiently small, then $\hat{D}$ is connected. When α is too large, ultra-high contrast regime, we determine a nearly optimal set which is an approximation of the minimal set for $\lambda(D)$. For the special case of a ball Ω centered at the origin we verify under symmetry conditions on ζ that $\Omega \setminus \hat{D}$ is also a ball centered at the origin. Section 4 is devoted to the numerical solution of the nonlinear eigenvalue and the shape optimization problem. We propose a numerical method for the solution of the nonlinear eigenvalue problem. The optimal configuration $\hat{D}$ is determined by a gradient type method. For this purpose we derive a formula for the shape derivative of the eigenvalue. The paper closes with some numerical examples and concluding remarks.

2. Existence result for optimization problem (1.2)

This section is devoted to prove the existence of a solution of problem (1.2). We take advantage of a variational characterization of the ground state energy which follows immediately from a generalization of the minmax characterization of the eigenvalues of Poincaré to eigenvalue problems depending nonlinearly on the eigenparameter given in [13–15].

Although the potential function is $\alpha\chi_D$, a variational formula is derived for a more general equation where the potential function is $v(x) \geq 0$ in $L^\infty(\Omega)$. Multiplying (1.1) by $\varphi \in H_0^1(\Omega)$ and integrating by parts, one gets the following variational formulation of (1.1): Find $\lambda \in \mathbb{R}$ and $u \in H_0^1(\Omega)$, $u \neq 0$ such that

$$a(\lambda, u, \varphi) := \int_\Omega \zeta(\lambda, x)\nabla u \cdot \nabla\varphi dx + \int_\Omega vu\varphi dx = \lambda \int_\Omega u\varphi dx =: \lambda b(u, \varphi), \tag{2.1}$$

for all φ in $H_0^1(\Omega)$.

We assume that for every $\lambda \geq 0$

$$\zeta(\lambda, \cdot) \in C(\bar{\Omega}) \quad \text{and} \quad \zeta(\lambda, .) \geq \theta_\lambda > 0. \tag{2.2}$$

Then the bilinear form $a(\lambda, \cdot, \cdot)$ is $H_0^1(\Omega)$-elliptic, continuous and symmetric. Further, $b(\cdot, \cdot)$ is a symmetric, completely continuous and positive definite bilinear form on $H_0^1(\Omega)$.