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Asymptotic behaviors of radially symmetric solutions to diffusion problems with Robin boundary condition in exterior domain[☆]

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A B S T R A C T

In this paper we study nonlinear diffusion problems of the form $u_t = \Delta u + f(u)$ with Robin boundary condition in exterior domain and heterogeneous environment where $f(u)$ is a bistable term. First we prove that the radially symmetric solution converges to its equilibrium locally uniformly in the exterior domain. Then we discuss the existence of some certain equilibrium and obtain a spreading–transition–vanishing trichotomy result. Finally the behavior changes with respect to the initial data are presented.

1. Introduction

We consider the diffusive equation

$$u_t = \Delta u + f(u) \qquad t > 0,\ |x| > 1,\ x \in \mathbb{R}^n \tag{1}$$

with Robin boundary condition $u(t,x) = b\frac{\partial u}{\partial n}(t,x)$ at $|x| = 1$ where b is a nonnegative constant. For simplicity, we assume that the environment and the solution are radially symmetric. Thus we will study the positive solution $u(t,r)(r = |x|)$ of the following problem

$$\begin{cases} u_t = u_{rr} + \dfrac{n-1}{r}u_r + f(u), & t > 0, r > 1, \\ u(t,1) = bu_r(t,1), & t > 0, \\ u(0,r) = u_0(r), & r \geqslant 1, \end{cases} \tag{2}$$

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where $f : [0, \infty) \to \mathbb{R}$ is a bistable nonlinearity satisfying

$$f(0) = f(\alpha) = f(1) = 0,\ f(u) \begin{cases} < 0 & \text{in } (0, \alpha), \\ > 0 & \text{in } (\alpha, 1), \\ < 0 & \text{in } (1, \infty), \end{cases} \quad f'(0) < 0,\ f'(1) < 0, \tag{3}$$

$$f'(\alpha) > 0,\ f(u) \le Ku,\ \int_0^\theta f(s)ds = 0,\ -\inf_{s>0} f(s)/s < \infty, \tag{4}$$

where $K > 0, \alpha \in (0, 1), \theta \in (\alpha, 1)$ are some constants. The initial function u_0 belongs to $\mathscr{X}(h_0)$ for some $h_0 > 1$, where

$$\mathscr{X}(h_0) := \Big\{ \phi \in C([1, \infty)) : \phi(1) = b\phi'(1); \phi \geqslant, \not\equiv 0 \text{ in } [1, h_0]; \phi \equiv 0 \text{ in } (h_0, \infty) \Big\}.$$

For Eq. (1) complemented by the homogeneous Dirichlet, Neumann or Robin boundary condition on $\mathbb{R}$ or on the half line, convergence of solutions of the problem to a special type of patterns (stationary states, traveling waves) in dependence on the initial data was studied by many authors [1–6] and the references therein. In [5] and [6] the authors gave a complete description of the asymptotic behavior of the solution for the one dimensional problem equipped with Neumann and Robin boundary condition respectively. They established a sharp threshold result: for any nontrivial $\phi \geqslant 0$ with compact support, there exists $\sigma^* > 0$ such that the solutions $u(t, \cdot; \sigma\phi)$ converge to zero when $\sigma < \sigma^*$ and the integral norm of solutions $u(t, \cdot; \sigma\phi)$ tend to infinity with growing time when $\sigma > \sigma^*$. Nevertheless, the behavior of the threshold trajectories is rather different for different types of boundary conditions. That is, the solution $u(t, \cdot; \sigma^*\phi)$ converges to a ground state with a finite shift for Neumann boundary condition case and with an infinite shift for Dirichlet boundary condition case, while for solution of problem (1) with Robin boundary condition both cases might happen depending on the value of b in (2).

The main purpose of this paper is to investigate the long time behavior of radially symmetric solutions of problem (2). The problems in multi-dimensions become more complicated than those in one space dimension. Even in the radially symmetric case where the problems could be transformed into one dimensional problems as in (2), there are many difficulties raised by the extra term $\frac{n-1}{r}u_r$ when the dimension $n \geqslant 2$. For instance, if $n = 1$ it is not difficult to give all kinds of stationary solutions by phase plane analysis (see [7]) which is failed for the case of $n \geqslant 2$ and we need different way to analyze the solutions of related elliptic problems. Actually we state several lemmas in Section 2 based on a type of shooting argument to present the existence and properties of stationary solutions.

It is easy to check by the comparison principle that $0 < u(t, r) \leqslant C$ for some constant C and $t > 0, r > 1$. We will show in the following first main result that the bounded nonnegative solution converges to the solution of its stationary problem.

Theorem 1. *Assume that f satisfies* (3) and (4). *Let u be a solution of* (2) *defined for all $t > 0$. Then $u(t, \cdot)$ converges, as $t \to \infty$, locally uniformly in $[1, \infty)$ to a nonnegative solution of the following elliptic problem*

$$\begin{cases} \Delta v + f(v) = 0, & r \in (1, \infty), \\ v(1) = bv'(1), \end{cases} \tag{5}$$

where $\Delta v = v'' + \frac{n-1}{r}v'$.

The above result could be proved as the argument of [5,8,9] with some modifications. For the convenience of reading we state it briefly in Section 3.

Now we consider a family of initial data and analyze the long time behavior of the solutions of (2).