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# Mathematical analysis of obstacle problems for pricing fixed-rate mortgages with prepayment and default options

Maria del Carmen Calvo-Garrido, Carlos Vázquez*

*Department of Mathematics, University of A Coruña. Campus Elviña s/n, 15071 A Coruña, Spain*

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ABSTRACT

In this paper, we address the mathematical analysis of a partial differential equation model for pricing fixed-rate mortgages with prepayment and default options, where the underlying stochastic factors are the house price and the interest rate. The mathematical model is posed in terms of a sequence of linked complementarity problems, one for each month of the loan life, associated with a uniformly parabolic operator. We study the existence of a strong solution to each one of the obstacle problems.



## 1. Introduction

A mortgage is a financial contract between two parts, a borrower and a lender, in which the borrower obtains funds from the lender (a bank or a financial institution, for example) by using a risky asset as a guarantee (collateral), usually a house. In this work, we focus on the mathematical model for fixed rate mortgages with monthly payments. The loan is reimbursed through monthly payments until the cancellation of the debt at maturity date. Thus, the mortgage value is understood as the discounted value of the future monthly payments (without including a possible insurance on the loan by the lender) and the underlying stochastic factors are the interest rate and the house price. The mathematical model is posed in [1,2] so that prepayment is allowed at any time during the life of the loan and default only can occur at any monthly payment date. Thus, the mathematical model is posed in terms of a sequence of linked complementarity problems associated to a parabolic partial differential equation, one for each month of the loan life. The link between obstacle problems comes from the condition at the end of the month defined as the mortgage value at this date obtained from the obstacle problem for next month. The existence of solution for each obstacle problem in this sequence is an open problem treated in the present paper. In [1,2] a log-normal process is assumed for house price evolution, so that this value evolves continuously whereas the interest rate dynamics

* Corresponding author.
*E-mail addresses:* mcalvog@udc.es (M.C. Calvo-Garrido), carlosv@udc.es (C. Vázquez).

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is described by means of the CIR model. More recently, in [3] a jump–diffusion model is proposed to describe the house price dynamics to account for bubble or crisis phenomena in real state markets. The numerical resolution of these problems has been addressed by using different techniques (see [1,2,4,5], for example).

The main objective of this paper is the mathematical analysis of the obstacle problems involved in the valuation of fixed rate mortgages with prepayment and default options. We propose an approach based on the concept of strong solution. The existence of a strong solution is studied in the framework of uniformly parabolic PDEs with variable coefficients, mainly adapting the results in [6]. The mathematical analysis of obstacle problems related to finance has been addressed in the literature. For example, in [7] the authors prove the existence of a strong solution for an obstacle problem linked to a non-uniformly parabolic operator of Kolmogorov type. This kind of Kolmogorov operators also appear in the complementarity problems that arise in the pricing of other financial products, such as American Asian options [8], pension plans with early retirement [9] or stock loans [10].

This paper is organized as follows. In Section 2 we pose the pricing model in terms of a sequence of complementarity problems. In Section 3 we first prove the existence of a unique strong solution to each obstacle problem in a bounded domain by adapting a penalization technique. Next, we prove the existence of a strong solution to the obstacle problem in the unbounded domain by solving a sequence of complementarity problems in regular bounded domains.

## 2. Mathematical modelling of the pricing problem

A mortgage can be treated as a derivative financial product, for which the underlying state variables are the house price and the term structure of interest rates. So, we have to model the dynamics of the underlying factors. Under risk neutral probability, the value of the house at time $t$, $H_t$, is assumed to follow the stochastic differential equation (see [11]):

$$dH_t = (r - \delta)H_t dt + \sigma_H H_t dX_t^H, \tag{1}$$

where $r$ is the interest rate, $\delta$ is the 'dividend-type' per unit service flow provided by the house, $\sigma_H$ is the house-price volatility and $X_t^H$ is the standard Wiener process associated to the house price. The other source of uncertainty, the stochastic interest rate $r_t$ at time $t$, is assumed to be a classical Cox–Ingersoll–Ross (CIR) process [12], satisfying

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r \sqrt{r_t} dX_t^r, \tag{2}$$

where $\kappa$ is the speed of adjustment in the mean reverting process, $\theta$ is the long term mean of the short-term interest rate (steady state spot rate), $\sigma_r$ is the interest-rate volatility parameter and $X_t^r$ is the standard Wiener process associated to the interest rate. Wiener processes, $X_t^H$ and $X_t^r$ can be assumed to be correlated according to $dX_t^H dX_t^r = \rho dt$, where $\rho$ is the instantaneous correlation coefficient.

Following the same notation as in [1], we assume that the mortgage is repaid by a sequence of monthly payments at dates $T_m$, $m = 1, \ldots, M$, where $M$ is the number of months of loan life. Assuming that $T_0 = 0$, let $\Delta T_m = T_m - T_{m-1}$ the duration of month $m$. Moreover, $\tau_m = T_m - t$ denotes the time until the payment date in month $m$, $c$ is the fixed contract rate and $P(0)$ is the initial loan (i.e. the principal at $T_0 = 0$), the fixed mortgage payment ($MP$) is given by:

$$MP = \frac{(c/12)(1 + c/12)^M P(0)}{(1 + c/12)^M - 1}. \tag{3}$$

For $m = 1, \ldots, M$, the unpaid loan just after the $(m - 1)$th payment date is

$$P(m - 1) = \frac{((1 + c/12)^M - (1 + c/12)^{m-1})P(0)}{(1 + c/12)^M - 1}. \tag{4}$$