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A note on deriving linearizing transformations for a class of second order nonlinear ordinary differential equations

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ABSTRACT

We present a method of deriving linearizing transformations for a class of second order nonlinear ordinary differential equations. We construct a general form of a nonlinear ordinary differential equation that admits Bernoulli equation as its first integral. We extract conditions for this integral to yield three different linearizing transformations, namely point, Sundman and generalized linearizing transformations. The explicit forms of these linearizing transformations are given. The exact forms and the general solution of the nonlinear ODE for these three linearizables cases are also enumerated. We illustrate the procedure with three different examples.

1. Introduction

Nonlinear ordinary differential equations (ODEs) can be solved in a number of ways. For example, integration by quadrature, exploring Darboux polynomials and/or Jacobi last multipliers, order reduction procedure through symmetry methods, direct linearization and so on [1–7]. Among the above, the linearization of nonlinear ODEs got impetus in recent years [8–15]. The task here is to transform the given nonlinear ODE into a linear ODE whose solution is known. Confining our attention on second order nonlinear ODEs a systematic study on this problem has been initiated by Sophus Lie long ago [4]. He demonstrated that the nonlinear ODEs that can be transformed to free particle equation should be cubic in first derivative whose coefficients (which are functions of t and x) should satisfy a couple of second order partial differential equations [4,5]. The underlying linearizing transformation (LT) is considered as point transformation (PT) since the new dependent (w) and independent (z) variables are functions of old dependent (x) and independent (t) variables alone, that is $w = F(t, x)$ and $z = G(t, x)$ [5]. Later it has been

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shown that the linearizable ODEs under point transformations admit maximal Lie point symmetries [5]. Subsequently linearizing PTs have been identified from the Lie point symmetries [4] itself.

The linearization of nonlinear ODEs under nonlocal transformation has also been investigated in detail [16–19]. The necessary and sufficient condition for a second order nonlinear ODE to be linearizable under the Sundman transformation (ST), $w = f(t,x)$, $z = \int g(t,x)dt$, had been analyzed by Duarte et al. [17]. Here f and g are functions of t and x only. Unlike the PT the new independent variable z is considered in nonlocal form. Since the independent variable z is nonlocal, it is difficult to invert the solution from its linear counterpart [5]. The connection between λ-symmetries and Sundman linearizable ODEs of second-order ODEs has been analyzed by few authors, see for example Refs. [20,21] and references therein.

Apart from the above two LTs, second-order nonlinear ODEs can also be linearized by generalized linearizing transformations (GLT), namely $w = f(t,x)$ and $z = \int g(t,x,\dot{x})dt$ [8,9,22]. The difference between ST and GLT is that in the latter the new independent variable z is generalized to contain the first derivative. Unlike the above two LTs the connection between GLT and symmetries is not known. However, it has been demonstrated that equations which cannot be linearized by PT and ST can be linearized by GLT [22].

It is clear from the above facts that in linearization besides identifying the LTs one has to device a suitable procedure to derive the general solution of the considered nonlinear ODE. In this context, recently a simple but powerful method of identifying the LTs has been introduced [8,9]. In this method the LTs have been identified from the first integral itself by rewriting it as a ratio of two perfect derivative functions. A perfect derivative function that appears in the numerator acts as the new dependent variable (w) and the function that appears as a perfect derivative in the denominator acts as the new independent variable (z), which in turn linearizes the nonlinear ODE. Interestingly, unlike the other methods, this procedure readily gives all the aforementioned LTs, namely PT, ST and GLT in a simple and straightforward manner [8,9]. This method also produced several new LTs in higher order ODEs which are previously unknown [8,9].

In the earlier works, the usefulness of this method has been demonstrated only for specific examples. In this work, we derive a general form of second order nonlinear ODE that can be linearized by the aforementioned LTs. We also derive the first integral for the considered nonlinear ODE. We then extract the conditions for this integral to yield PT. We also present the explicit form of the PT that linearizes the nonlinear ODE. We then move on to identify the condition for this integral to give the ST. The explicit form of the ST that linearizes the nonlinear ODE is also given. Finally, by rewriting the integral suitably we explore the explicit form of the GLT that linearizes yet another family of ODEs. Our result shows that in the case of Sundman linearizable, the first integral should be a linear polynomial in $\dot{x}$. This result agrees with the one reported earlier [20]. In other words once a nonlinear ODE is identified as a linearizable equation then one can follow the procedure given in this paper and identify the LT readily. In the earlier works, the general solution of the nonlinear ODE has been derived only from the known solution of its linear counterpart which pose some obstacles in the case of ST and GLT since the independent variable in them are nonlocal. Differing from that, in this work, the general solution of the nonlinear ODE is given explicitly for all three cases. The general solution is derived by integrating the integral directly. The procedure developed in this paper not only gives the LTs but also its general solution in a straightforward way.

The structure of the paper is as follows: In Section 2, we identify the general form of the nonlinear ODE that admits Bernoulli equation as its first integral. In Section 3, we present the method of obtaining LTs from the first integral. We also derive the conditions to obtain PT and ST from this integral. The explicit forms of these two LTs are given. We then extract the explicit form of the GLT from the integral. In Section 4, we consider three different examples, one for PT, second for ST and last one for GLT, and demonstrate the method of identifying the LTs, first integral and the general solution. Finally, in Section 5, we present our conclusion.