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Remarks about a fractional Choquard equation: Ground state, regularity and polynomial decay

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ABSTRACT

With appropriate hypotheses on the nonlinearity f, we prove the existence of a ground state solution u for the problem

$$\left\{(-\Delta_p)^s u + A|u|^{p-2}u = \left(\frac{1}{|x|^\mu} * F(u)\right) f(u) \quad \text{in } \mathbb{R}^N,\right.$$

where $0 < \mu < N$, $(-\Delta_p)^s$ stands for the (s,p)-Laplacian operator, F is the primitive of f and A is a positive constant. When $\mu < p$, we also show that $u \in L^\infty(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$ and has polynomial decay.

1. Introduction

According to Lieb [24], the Choquard equation

$$-\Delta u + u = \left(\frac{A}{|x|} * |u|^2\right) u \quad \text{in } \mathbb{R}^3 \tag{1}$$

(A is a constant) was introduced by Ph. Choquard in 1976 to model a one-component plasma, despite being also related to previous works of H. Fröhlich and S. Pekar.

Since then, variations of the Choquard equation describe a series of phenomena. For example, when the attractive interaction of particles is weaker and has a continuing effect that endures more than that of the

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nonlinear Schrödinger equation, the evolution equation $i\partial_t\phi = \Delta\phi + (V * |\phi|^2)\phi$ models the interaction of large system of non-relativistic bosonic atoms and molecules.

The existence of non-trivial solutions of (1) was considered by Lions [26,27,30]: a solution $0 \neq u \in H^1(\mathbb{R}^3)$ was obtained by using critical point theory. Ma and Zhao in [29] obtained some qualitative properties of the positive solutions considering power like $|u|^q$. Considering the more general equation

$$-\Delta u + u = (I_\alpha * F(u))\, F'(u) \quad \text{in} \quad \mathbb{R}^N$$

(I_α denotes a Riesz potential), Moroz and Van Schaftingen [32,33] (see also [18]) proved the existence of a ground state and studied its symmetry and regularity properties.

When A is a coercive potential, (1) was treated via variational methods in [10,42], where existence of standing wave solutions and ground state solutions were obtained, respectively. In [1], by using the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [43], Ackermann studied the equation $-\Delta u + Vu = (W * u^2)u$ when V is a periodic potential and W belongs to a class of odd functions. In [2], Alves and Yang treated a generalized Choquard equation. A good survey of the Choquard equation can be found in [31].

The fractional (s,p)-Laplacian operator, $(-\Delta_p)^s$, arises in various fields, mainly when $p = 2$, such as continuum mechanics, phase transition phenomena, population dynamics, game theory and financial mathematics, see [4,9]. Concerning existence, non-existence and regularity results, see, e.g., [8,36–39]. Recently, the Choquard equation involving a fractional Laplacian operator has been studied extensively: see [12,13,41] (and references therein) for several ground state existence results when A is a constant. In [11] is treated a non-autonomous Choquard equation involving a fractional Laplace operator.

We refer to [6,7,17,19–21,23,25,34] for some existence and regularity results involving the operator $(-\Delta_p)^s$.

In this paper we consider a Choquard-type equation in a fractional (s,p)-Laplacian setting:

$$(-\Delta_p)^s u + A|u|^{p-2}u = \left(\frac{1}{|x|^\mu} * F(u)\right) f(u) \quad \text{in } \mathbb{R}^N, \tag{2}$$

where A is a positive constant, $0 < \mu < N$, $F(t) = \int_0^t f(r)\mathrm{d}r$ and $(-\Delta_p)^s$ denotes the (s,p)-Laplacian operator, defined by (see also Section 2)

$$(-\Delta_p)^s u(x) = 2 \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\,\mathrm{d}y.$$

Changing $(-\Delta_p)^s$ for $-\Delta$, in [40] Souto and de Lima have considered the potential $A = A(y,z)$, imposing a uniformly coercivity condition on the variable y.

In (2), we assume that f is a C^1 function, positive on $(0,\infty)$, that satisfies

(f1) $\lim_{t\to 0} \frac{|f(t)|}{t^{p-1}} = 0$;

(f2) $\lim_{t\to\infty} \frac{f(t)}{t^{q-1}} = 0$ for some $p < q < \frac{p_s^*}{2}\left(2 - \frac{\mu}{N}\right)$, where

$$p_s^* = \frac{Np}{N-sp} \quad \text{with} \quad sp < N;$$

(f3) $f'(t)t^2 - (p-1)f(t)t > 0$ for all $t > 0$.

An example of a function f satisfying these hypotheses is given by

$$f(t) = |t|^{q_1-1}t^+ + |t|^{q_2-1}t^+,$$

where $p < q_1 < q_2 < (1-\mu/N)p_s^*$ and $t^+ = \max\{t,0\}$. In this paper, we prove existence of a ground state solution, regularity and polynomial decay. We summarize our results: