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Extrema of the dynamic pressure in a solitary wave

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ABSTRACT

We study the dynamic pressure in an irrotational solitary wave propagating at the surface of water over a flat bed, under the influence of gravity. We consider the nonlinear regime, that is, the case of waves of moderate to large amplitude. We prove that, independently of the wave amplitude, the maximum of the dynamic pressure is attained at the wave crest, while its minimum is attained at infinity.

1. Introduction

Solitary waves can propagate on a free surface of water over a flat bed under the influence of gravity over long distances, while maintaining a constant shape. They are two-dimensional objects in the sense that they present essentially no variations in the horizontal direction perpendicular to the direction of propagation of the wave. They are thus fully characterised by a vertical cross section parallel to the direction of propagation, where they take the form of a single hump of elevation of the water, moving at constant speed. In the reference frame moving with the wave, their profile is steady and symmetric with respect to the vertical axis through the wave crest, and decreases rapidly away from this axis, see Fig. 1. The first observation of this phenomenon was made by Scott Russell in 1834 in a canal near Edinburgh. His report and subsequent laboratory experiments played an important role in the early developments of water waves theory, see [12].

We are interested here in the description of solitary waves of moderate and large amplitude, which fail to be captured accurately by the linear theory of water waves. As the effects of surface tension and viscosity are negligible in this regime, and it is physically reasonable to assume the water has a constant density ρ (we take $\rho = 1$ throughout), the waves are described by the Euler equation for a homogeneous incompressible fluid with free boundary, under the influence of gravity only. Moreover, motivated by the observation that large areas of abyssal plains in the oceans are essentially flat, we will restrict our attention here to a body of water over a flat bed.

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The rigorous mathematical investigation, beyond the linear theory approximation, of the fluid motion beneath gravity water waves has made important progress in recent years, see [7,3,9,8] and references in these papers. The study of the pressure in the fluid is of particular interest, from a theoretical point of view but also due to its important practical applications in maritime engineering. A good knowledge of the pressure field is indeed essential to compute the forces acting on maritime structures. On the other hand, pressure measurements at the bottom of the water can also be used to infer precious information about the waves on the surface [5,4,19,16,15].

In the context of irrotational waves, the pressure field was investigated by Constantin and Strauss [9] for periodic waves, and Constantin, Escher and Hsu [8] for solitary waves. Their main results concern the monotonicity properties of the pressure in the fluid domain. Although fluid motion can be driven by pressure gradients, in water at rest the *hydrostatic pressure* – which has a constant vertical gradient pointing downwards – only counterbalances gravity and does not induce any motion. The study of the relation between fluid motion and pressure therefore benefits from introducing the *dynamic pressure*, which is defined as the difference between the total pressure in the fluid and the hydrostatic pressure, see (11). The dynamic pressure beneath irrotational periodic gravity water waves was recently investigated by Constantin in [6], where it is proved that the maximum of the dynamic pressure is attained at the wave crest and its minimum at the wave trough. Since it is known [1] that periodic waves converge to solitary waves in the long-wave limit, it is natural to guess that similar results hold for solitary waves. (Note that one should be careful when applying this kind of reasoning to dynamic properties of the waves, for it was shown in [7] that particle trajectories in the fluid undergo a dramatic qualitative change in the long-wave limit.) In the present paper we prove, using maximum principles for elliptic partial differential equations, that it is indeed the case. Namely, the maximum of the dynamic pressure in an irrotational solitary wave is attained at the crest, while its minimum is attained at infinity.

2. Mathematical formulation of the problem

Irrotational solitary gravity water waves are two-dimensional. It was indeed proved in [10] that, in the absence of vorticity, no truly three-dimensional solitary waves can exist. A travelling solitary wave is thus fully characterised by the description of a cross section of the flow, perpendicular to the crest line. We choose Cartesian coordinates (X, Y), the Y-axis pointing vertically upwards, the X-axis being parallel to the direction of propagation of the wave. We require the flow to be at rest for $X \to \pm\infty$, and we choose the Y coordinate so that $Y = 0$ there, with the flat bed lying at depth $Y = -d,\ d > 0$. We suppose that the crest of the wave is at $X = 0$ at time $t = 0$.

We investigate the dynamic pressure in a permanent wave with profile $Y = \eta(X - ct)$, moving at constant speed $c > 0$, so we assume that the velocity field of the flow has the form

$$(u, v) = (u(X - ct), v(X - ct)).$$

Under these assumptions, time can be removed from the governing equations by describing the wave in the moving frame, that is, in the coordinates

$$x = X - ct, \qquad y = Y.$$

In the new reference frame, which moves at speed c in the direction of propagation of the wave, the wave is stationary and the flow is steady, see Fig. 1.

For water waves, it is physically reasonable to assume that the fluid is incompressible and homogeneous (with constant density $\rho = 1$), which yields the continuity equation

$$u_x + v_y = 0. \tag{1}$$