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On the covering of a Hill's region by solutions in systems with gyroscopic forces

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ABSTRACT

Consider a Lagrangian system with the Lagrangian containing terms linear in velocity. By analogy with the systems in celestial mechanics, we call a bounded connected component of the possible motion area of such a system a Hill's region. Suppose that the energy level is fixed and the corresponding Hill's region is compact. We present sufficient conditions under which any point in the Hill's region can be connected with its boundary by a solution with the given energy. The result is illustrated by examples from mechanics.

1. Introduction

Consider a mechanical system with the following Lagrangian function

$$L = L_2 + L_1 + L_0 = \frac{1}{2}A(q)\dot{q}\cdot\dot{q} + b(q)\cdot\dot{q} + V(q), \quad q \in M. \tag{1}$$

Here M is a smooth manifold and q denotes the local coordinates on M. Suppose that L_2 is a positive definite quadratic form, L_1 is defined by a 1-form on TM. Below, we assume that all considered objects are smooth (i.e., C^∞). The dynamics of the system is described by the equation $[L] = 0$, where $[\cdot]$ is a Lagrangian derivative. Since the total energy $L_2 - L_0 = h$, where $h \in \mathbb{R}$, does not change along the solutions of the considered system and $L_2 \geqslant 0$, then for any solution we have $V(q) + h \geqslant 0$. Therefore, for a given energy h, the possible motion area B_h is defined as follows

$$B_h = \{q \in M\colon V(q) + h \geqslant 0\}. \tag{2}$$

For the planar restricted circular three-body problem, B_h is usually called a Hill's region [3]. Similarly, for the Lagrangian system (1), we would call any connected component of B_h a Hill's region. In the following, we use the same notation B_h for any connected component of the possible motion area.

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If there are no gyroscopic terms in the Lagrangian function of the considered system, i.e., $L_1 \equiv 0$, the trajectories of motion in B_h are the paths of extremal length in the Jacobi metric $(h+V)\langle\cdot,\cdot\rangle$; where $\langle\cdot,\cdot\rangle$ is the kinetic metric $\langle\dot q,\dot q\rangle = 2L_2$. Note that the Jacobi metric is degenerate at the boundary ∂B_h which complicates the direct application of results from Riemannian geometry. Nevertheless, the following result concerning the covering of a Hill's region by solutions is proved in [9]:

Theorem 1.1 *([9]). If B_h is compact and there are no critical points of L_0 in ∂B_h, then any point $q \in B_h$ can be connected with some point of ∂B_h by a geodesic of the Jacobi metric.*

Since the work of Seifert [13] on closed geodesics, the case $L_1 \equiv 0$ has been intensively studied from different points of view. In particular, various results concerning periodic librations have been obtained (see, e.g. [1,12,5,6,10,8,7]).

The presence of gyroscopic terms in the Lagrangian makes the system "non-reversible" and may significantly change the qualitative behaviour of solutions. However, gyroscopic forces naturally appear in various mechanical systems. For instance, they appear after Routh reduction in systems with symmetry, in the presence of magnetic forces (the Lorentz force), or due to the use of a rotating reference frame. The trajectories of solutions of a system with gyroscopic forces are the extrema of the abbreviated action functional (or the Maupertuis action) for paths with a fixed energy h [2]:

$$F^* = \int_\gamma \frac{\partial L}{\partial \dot q}\dot q(\tau)\,d\tau. \tag{3}$$

Below we present a simple example of a system with arbitrarily small gyroscopic forces such that Theorem 1.1 is no longer true and B_h is not fully covered by the solutions, i.e., for at least one point in the possible motion area there is no extremum of F^* connecting the considered point with boundary ∂B_h.

In our work we consider system (1) in the general case of non-zero form L_1 and present sufficient conditions for the covering of B_h by solutions starting with zero velocities from its boundary ∂B_h. The result is illustrated on the planar restricted circular three-body problem with additional gyroscopic forces.

2. Some simple examples of systems with gyroscopic forces

In this section we shortly consider a simple integrable system with $L_1 \not\equiv 0$ and study qualitatively the effect of the gyroscopic forces on the covering of its Hill's region. First, we give the definition of the covering.

Definition 2.1. Let B_h be a Hill's region of the system (1). We say that B_h is covered by the solutions starting at ∂B_h, if for any $q_b \in B_h$ there exists a point $q_a \in \partial B_h$ and a solution $q\colon [a,b] \to M$ of the system (1) such that $q(a) = q_a$, $q(b) = q_b$ and the total energy $L_2 - L_0$ along this solution is equal to h.

As it was mentioned above, in the absence of gyroscopic forces, a Hill's region is always covered by the solutions starting at its boundary. Let us show that an arbitrary small gyroscopic force may lead to the appearance of an uncovered area inside a Hill's region.

Consider the system of a massive point moving on a plane under the action of a potential force and the Lorentz force. We suppose that in the standard polar coordinates the Lagrangian has the form

$$L = \frac{1}{2}(\dot r^2 + r^2\dot\varphi^2) + \omega(r) r^2 \dot\varphi - \frac{r^2}{2}. \tag{4}$$

Here ω is a smooth function such that $\omega'(0) = \omega'(1) = 0$ and $|\omega| < \alpha$, $\alpha \in \mathbb{R}$.

For a given energy h, the Hill's region B_h is described by the inequality $r \leqslant \sqrt{2h}$. At initial time, we have $r(0) = r_0 = \sqrt{2h}$ and $\dot r(0) = 0$. Using the first integral corresponding to the cyclic coordinate φ, one can