Nonlinear Analysis 154 (2017) 189–209

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

A density problem for Sobolev spaces on Gromov hyperbolic domains

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ARTICLE INFO

Article history:
Received 26 May 2016
Accepted 13 July 2016
Communicated by Enzo Mitidieri

Dedicated to Nicola Fusco

MSC:
46E35

Keywords:
Sobolev space
Density

ABSTRACT

We prove that for a bounded domain $\Omega \subset \mathbb{R}^n$ which is Gromov hyperbolic with respect to the quasihyperbolic metric, especially when Ω is a finitely connected planar domain, the Sobolev space $W^{1,\infty}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for any $1 \leq p < \infty$. Moreover if Ω is also Jordan or quasiconvex, then $C^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\Omega)$ for $1 \leq p < \infty$.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain with $n \geq 2$. We denote by $D_i u = \frac{\partial u}{\partial x_i}$ the (weak) ith partial derivative of a locally integrable function u, and by $\nabla u = (D_1 u, \ldots, D_n u)$ the (weak) gradient. Then for $1 \leq p \leq \infty$ we define the Sobolev space as

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid D_i u \in L^p(\Omega),\ 1 \leq i \leq n\},$$

with the norm

$$\|u\|^p_{W^{1,p}(\Omega)} = \int_\Omega |u(x)|^p + |\nabla u(x)|^p \, dx$$

for $1 \leq p < \infty$, and

$$\|u\|_{W^{1,\infty}(\Omega)} = \operatorname*{esssup}_{x \in \Omega} |u(x)| + \sum_{1 \leq i \leq n} \operatorname*{esssup}_{x \in \Omega} |D_i u(x)|.$$

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http://dx.doi.org/10.1016/j.na.2016.07.007
0362-546X/

It is a fundamental property of Sobolev spaces that smooth functions defined in Ω are dense in $W^{1,p}(\Omega)$ for any domain $\Omega \subset \mathbb{R}^n$ when $1 \le p < \infty$. If each function in $W^{1,p}(\Omega)$ is the restriction of a function in $W^{1,p}(\mathbb{R}^n)$, one can then obviously use global smooth functions to approximate functions in $W^{1,p}(\Omega)$. This is in particular the case for Lipschitz domains. Moreover, if Ω satisfies the so-called "segment condition", then one has that $C^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\Omega)$; see e.g. [1,15] for references.

In the planar setting, Lewis proved in [12] that $C^\infty(\mathbb{R}^2)$ is dense in $W^{1,p}(\Omega)$ for $1 < p < \infty$ provided that Ω is a Jordan domain. More recently, in [8] it was shown by Giacomini and Trebeschi that, for bounded simply connected planar domains, $W^{1,2}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for all $1 \le p < 2$. Motivated by the results above, Koskela and Zhang proved in [11] that for any bounded simply connected domain and any $1 \le p < \infty$, $W^{1,\infty}(\Omega)$ is dense in $W^{1,p}(\Omega)$, and $C^\infty(\mathbb{R}^2)$ is dense in $W^{1,p}(\Omega)$ when Ω is Jordan.

In this paper, we extend the main idea in [11] so as to handle both multiply connected and higher dimensional settings. It turns out that simply connectivity (or trivial topology) is not sufficient for approximation results in higher dimensions.

Theorem 1.1. *Given $1 < p < \infty$, there is a bounded domain $\Omega \subset \mathbb{R}^3$, homeomorphic to the unit ball via a locally bi-Lipschitz homeomorphism, such that $W^{1,q}(\Omega)$ is not dense in $W^{1,p}(\Omega)$ for any $q > p$.*

Recall that $f : \Omega \to \Omega'$ is locally bi-Lipschitz if for every compact set $K \subset \Omega$ there exists $L = L(K)$ such that for all $x, y \in K$

$$\frac{1}{L}|x-y| \le |f(x) - f(y)| \le L|x-y|.$$

The above example shows that the planar setting is very special. The crucial point is that a simply connected planar domain is conformally equivalent (by the Riemann mapping theorem) to the unit disk, and conformal equivalence is in general much more restrictive than topological equivalence. One could then ask if the planar approximation results extend to hold for those spatial domains that are conformally equivalent to the unit ball. This is trivially the case since the Liouville theorem implies that such a domain is necessarily a ball or a half-space. A bit of thought reveals that bi-Lipschitz equivalence is also sufficient. Our results below imply that bi-Lipschitz equivalence can be relaxed to quasiconformal equivalence to the unit ball or even to quasiconformal equivalence to a uniform domain, a natural class of domains in the study of (quasi)conformal geometry.

In order to state our main result, we need to introduce some terminology.

Definition 1.2. Let $\Omega \subsetneqq \mathbb{R}^n$ be a domain. Then the associated quasihyperbolic distance between two points $z_1, z_2 \subset \Omega$ is defined as

$$\operatorname{dist}_{qh}(z_1, z_2) = \inf_\gamma \int_\gamma \operatorname{dist}(z, \partial\Omega)^{-1}\, dz,$$

where the infimum is taken over all the rectifiable curves $\gamma \subset \Omega$ connecting z_1 and z_2. A curve attaining this infimum is called a quasihyperbolic geodesic connecting z_1 and z_2. The distance between two sets is also defined in a similar manner.

Moreover, a domain Ω is called δ-Gromov hyperbolic with respect to the quasihyperbolic metric, if for all $x, y, z \in \Omega$ and any corresponding quasihyperbolic geodesics $\gamma_{x,y}, \gamma_{y,z}, \gamma_{x,z}$, we have

$$\operatorname{dist}_{qh}(w, \gamma_{y,z} \cup \gamma_{x,z}) \le \delta,$$

for any $w \in \gamma_{x,y}$.

For the existence of quasihyperbolic geodesics we refer to [4, Proposition 2.8]. For applications, it is usually easier to apply one of the equivalent definitions, see Lemma 2.1. Recall that a set $E \subset \mathbb{R}^n$ is called