Economics Letters 159 (2017) 128–133

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

Behavior of the standard Dickey–Fuller test when there is a Fourier-form break under the null hypothesis[☆]

Lixiong Yang[a,*], Chingnun Lee[b], Jen-Je Su[c]

[a] School of Management, Lanzhou University, Lanzhou, China
[b] Institute of Economics, National Sun Yat-sen University, Kaohsiung, Taiwan
[c] Department of Accounting Finance and Economics, Griffith University, Brisbane, Australia

HIGHLIGHTS

- We assume that the null unit root process is with a Fourier component.
- We derive the asymptotic distribution of the standard Dickey–Fuller (DF) test.
- Asymptotic distributional results generate interesting predictions.
- The converse Perron phenomenon may occur when a Fourier-form break exists.
- The predictions are confirmed by simulation results.

ARTICLE INFO

Article history:
Received 25 February 2017
Received in revised form 21 June 2017
Accepted 14 July 2017
Available online 29 July 2017

JEL classification:
C12
C15

Keywords:
Structural break
Fourier approximation
Unit root test

ABSTRACT

We derive the null asymptotic distribution of the standard Dickey–Fuller test with the existence of an unnoticed Fourier component. The so-called converse Perron phenomenon might occur, but only in the trend-case with a low-frequency Fourier component and small error variance.

1. Introduction

Following the seminal work of Perron (1989), it is now a well-known fact that the usual Dickey–Fuller (DF) test is inconsistent when applied to stationary series with a break. In contrast, Leybourne et al. (1998) and Leybourne and Newbold (2000) illustrate a "converse Perron phenomenon", suggesting that the usual DF test tends to suffer enormous size distortion when applied to a unit root process with an abrupt break (particularly, if the break occurs early in the sample). Accordingly, the usual DF test is likely to mix up a stationary series carrying a break with a unit root process and mistake a unit root process with a break for a stationary series.

Since 1989, a vast literature has developed around incorporating breaks into unit root testing. The literature begins with considering a single exogenous break (i.e. a break at a known point) and steadily evolves into permitting for possible multiple endogenous breaks (i.e. breaks at unknown dates); see Perron (2006) for a comprehensive survey. In practice, this line of research requires to assume the maximum number of breaks and identify the break dates. These parameters are crucial to the performance of break-adjusted unit root tests but they are hard to be properly estimated.

Becker et al. (2006) and Enders and Lee (2012a, b) suggest a new approach in handling breaks for unit root tests. They demonstrate that the flexible Fourier expansion of Gallant (1981) can well approximate the deterministic component of an economic time series with numerous breaks. The new approach is advantageous for its simplicity as commonly only a single frequency is sufficient to achieve a reasonable approximation. In terms of empirical relevance, according to Enders and Lee (2012a, b), using the specific

☆ We thank an anonymous referee and the editor for very valuable comments and suggestions. Remaining errors are our own.

* Correspondence to: School of Management, Lanzhou University, 222 South Tianshui Road, Lanzhou, Gansu, 730000, China
E-mail address: ylx@lzu.edu.cn (L. Yang).

http://dx.doi.org/10.1016/j.econlet.2017.07.016
0165-1765/

frequency $k = 1$ often leads to a good approximation for breaks of unknown form in economic series.

According to Enders and Lee (2012a), ignoring Fourier-type breaks in a stationary series can lead to a Perron-like phenomenon of inconsistency. However, the literature is silent if a similar converse Perron phenomenon will occur when a unit root process with a Fourier component is considered. The main purpose of this paper is to fill this gap in the literature. To this end, we derive the asymptotic distribution of the DF t-statistic under the null hypothesis, assuming the null unit root process is accompanied with a Fourier component. Interestingly, we find that ignoring a Fourier component will end up with very different outcomes: the null hypothesis can be either over-rejected or under-rejected, depending on the setting of the Fourier component, the variance of the disturbance, and whether the DF test allows for a linear trend. In other words, the converse Perron phenomenon can arise, but only in certain cases. All the results in this note are derived by assuming the integer frequency k and "$\Rightarrow$" stands for weak convergence.

2. The standard DF test under a Fourier-form break

Let y_t be generated by the following $AR(1)$ model

$$(1-\phi L)(y_t-\alpha(t)-\gamma t)=u_t,\quad t=1,2,\ldots,T, \tag{1}$$

where u_t is an $i.i.d.$ disturbance with zero mean and constant variance σ^2, $\alpha(t)$ is a time-varying deterministic break function, and γt is a linear deterministic trend. The initial value y_0 is assumed to be $O(1)$. Following Enders and Lee (2012a, b) and Lee et al. (2016), $\alpha(t)$ is set to the following single-frequency Fourier form:[1]

$$\alpha(t)=\alpha_0+\beta_1\sin(2\pi kt/T)+\beta_2\cos(2\pi kt/T), \tag{2}$$

where β_1 and β_2 measure the amplitude and displacement of sinusoidal components and k represents a particular frequency.

We are of interest to test for a unit root ($\phi = 1$) against stationarity ($\phi < 1$) from the standard DF test. Specifically, we aim to examine the situation under the unit root null hypothesis when the Fourier component ($\alpha(t)$) in (1) is unnoticed. Similar to Leybourne et al. (1998) and Leybourne and Newbold (2000), we assume the magnitude of the break amplitude parameters, β's, is proportional to $T^{1/2}$. This assumption ensures that, asymptotically, the break component $\alpha(t)$ and the random walk component of y_t are of the same order of magnitude (in probability). Thus, the asymptotic distribution of the test statistics depends on the break.

Theorem 1. *Suppose y_t is generated based on (1) and (2) with $\phi = 1$ and assume $\beta_1 = \kappa_1 T^{1/2}$ and $\beta_2 = \kappa_2 T^{1/2}$, where κ_1 and κ_2 are constants. We consider the following two DF tests.*

***(a). Trend case**: Let $t^{DF_t,B}$ be the standard DF t-test statistics of the regression: $\Delta y_t = \rho y_{t-1} + c_1 + c_2 t + e_t$. We have*[2] Eq. (3) is given in Box I, *in which*

$$\boldsymbol{q}=\begin{bmatrix} W(1) \\ W(1)-\int_0^1 W(r)dr \end{bmatrix},\qquad \boldsymbol{h}=\begin{bmatrix} \int_0^1 W(r)dr \\ \int_0^1 rW(r)dr \end{bmatrix},$$

$$\boldsymbol{\Psi}=\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix},$$

$$\mathrm{d}_1=\int_0^1 W(r)dr+2\pi k\int_0^1 \sin(2\pi kr)\left[\int_0^r W(s)ds\right]dr,$$

$$\mathrm{d}_2=-2\pi k\left(\int_0^1 \cos(2\pi kr)\left[\int_0^r W(s)ds\right]dr\right) -\frac{3}{\pi k}\left(\int_0^1 W(r)dr-2\int_0^1 rW(r)dr\right),$$

$$\mathrm{d}_3=-2\pi k\int_0^1 \cos(2\pi kr)W(r)dr-\frac{3}{\pi k}\left[2\int_0^1 W(r)dr-3W(1)\right],$$

$$\mathrm{d}_4=W(1)+2\pi k\int_0^1 \sin(2\pi kr)W(r)dr.$$

***(b). Mean case**: If $\gamma = 0$ in (1), and let $t^{DF_c,B}$ be the standard DF t-test statistics of the regression: $\Delta y_t = \rho y_{t-1} + c_1 + e_t$, then* Eq. (4) *is given in* Box II, *in which $\mathrm{d}_2^* = -2\pi k\int_0^1 \cos(2\pi kr)\left[\int_0^r W(s)ds\right]dr$ and $\mathrm{d}_3^* = -2\pi k\int_0^1 \cos(2\pi kr)W(r)dr$.*

Proof. See Appendix A. ∎

Clearly, the asymptotic distribution of the DF statistic depends on σ, k, κ_1 and κ_2 except for the special case of no breaks ($\kappa_1 = \kappa_2 = 0$).[3] Hence, ignoring the Fourier component may lead to non-trivial size distortion. However, as the asymptotic distribution is complicated, it is hard to quantify the extent of inconsistency. To focus on the impact of the size when the Fourier components are ignored, following Leybourne and Newbold (2000), further insight can be shed by considering the extreme case of large κ's. For large κ_1 and κ_2, as $T \to \infty$, then approximately[4]

$$t^{DF_t,B}\to\frac{\frac{6\kappa_1\kappa_2}{\pi k\sigma}}{\left(\frac{1}{2}(\kappa_1^2+\kappa_2^2)-\frac{3\kappa_1^2}{(\pi k)^2}\right)^{1/2}},\quad \text{(Trend)} \tag{5}$$

and[5]

$$t^{DF_c,B}\to 0.\quad \text{(Mean)} \tag{6}$$

The result of (6) implies that, in the mean case, as $t^{DF_c,B} \to 0$, the DF test is likely to under-reject the null hypothesis when κ's are large. On the other hand, for the trend case, since the limiting distribution of $t^{DF_t,B}$ in (5) hinges on k, κ_1, κ_2 and σ, the direction of inconsistency is unclear. We plot the limiting distribution in (5) with two illustrative examples: Fig. 1(a) for $\kappa_1 = 2$ and $\kappa_2 = -2$ and Fig. 1(b) for $\kappa_1 = \kappa_2 = 2$, under various combinations of σ and k. Fig. 1(a) shows that, when κ_1 and κ_2 are large with opposite signs and $\kappa\cdot\sigma$ is small, $t^{DF_t,B}$ converges to a low negative value and the null hypothesis is likely to be over-rejected. However, the chance of over-rejection lessens as $k\cdot\sigma$ increases. Conversely, according to Fig. 1(b), when κ_1 and κ_2 are with the same sign, since $t^{DF_t,B}$ converges to a positive value, under-rejection is likely to occur for any combination of σ and k.

3. Simulation evidence

In this section, we examine the performance of the DF test using

1 Without loss of generality, it is assumed that $\alpha(0) = 0$.

2 In Appendix A, we have shown that (from (A.7)), for large κ's, the bias of the estimator of σ^2 is of the same order of magnitude (in probability) with the term, $\frac{1}{2}\left\{\frac{(2\pi k)^2\kappa_1^2}{T}+\frac{(2\pi k)^2\kappa_2^2}{T}\right\}$. Hence, for a small σ^2 and large κ's, the bias is considerable in finite sample size. It therefore needs a large sample size T to obtain this asymptotic result under this circumstance.

3 When $\kappa_1 = \kappa_2 = 0$, the distribution shrinks to the usual DF distribution. For example, $t^{DF_t,B} \Rightarrow \frac{\int_0^1 W(r)dW(r)-\boldsymbol{q}'\boldsymbol{\Psi}^{-1}\boldsymbol{h}}{\int_0^1 W^2(r)dr-\boldsymbol{h}'\boldsymbol{\Psi}^{-1}\boldsymbol{h}}$.

4 Simulation results show that d_1, d_2, d_3 and d_4 in (3) are largely symmetric around zero and mostly lie between −0.1 and 0.1. Therefore, they are negligible when κ's are large.

5 When κ's are large, $\frac{1}{2}(\kappa_1^2+\kappa_2^2)$ in the denominator of (4) dominates all other terms and, as a result, $t^{DF_c,B}$ converges to zero. Simulation results also show that d_2^* and d_3^* are symmetric around zero and, for most cases, less than 10^{-5} in magnitude.