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Hyperbolic discounting of the far-distant future

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ABSTRACT

We prove an analogue of Weitzman's (1998) famous result that an exponential discounter who is uncertain of the appropriate exponential discount rate should discount the far-distant future using the lowest (i.e., most patient) of the possible discount rates. Our analogous result applies to a hyperbolic discounter who is uncertain about the appropriate hyperbolic discount rate. In this case, the far-distant future should be discounted using the probability-weighted *harmonic mean* of the possible hyperbolic discount rates.

1. Introduction

Consider an individual – or Social Planner – who ranks streams of outcomes over a continuous and unbounded time horizon $T = [0, \infty)$ using a discounted utility criterion with discount function $D : T \to (0, 1]$. We assume throughout that D is differentiable, strictly decreasing and satisfies $D(0) = 1$. For example, D might have the *exponential* form

$$D(t) = e^{-rt}$$

for some constant *discount (or time preference) rate*, $r > 0$. Such discounting may be motivated by suitable preference axioms (Harvey, 1986) or as a survival function with constant hazard rate, r (Sozou, 1998). For an arbitrary (i.e., not necessarily exponential) discount function, Weitzman (1998) defines the *local (or instantaneous) discount rate*, $r(t)$, using the relationship:

$$D(t) = \exp\left(-\int_0^t r(\tau)d\tau\right) \quad \Leftrightarrow \quad r(t) = -\frac{D'(t)}{D(t)}. \tag{1}$$

Note that $r(t)$ is constant if (and only if) D has the exponential form.

Weitzman (1998) considers a scenario in which the decision-maker is uncertain about the appropriate discount function to use. She may, for example, be uncertain about the true (constant) hazard rate of her survival function, as in Sozou (1998). The decision-maker entertains n possible scenarios corresponding to n possible discount functions D_i, $i = 1, 2, \ldots, n$, with associated local discount rate functions r_i. Suppose that scenario i has probability $p_i > 0$, with $\sum_{i=1}^n p_i = 1$, and that the decision-maker discounts according to the *expected (or certainty equivalent)* discount function

$$D = \sum_{i=1}^n p_i D_i. \tag{2}$$

Such a discount function may also arise if the decision-maker is a utilitarian Social Planner for a population of n heterogeneous individuals, as in Jackson and Yariv (2015).

Let r be the local discount rate function associated with certainty equivalent discount function (2). Weitzman (1998) studies the limit behaviour of $r(t)$ as $t \to \infty$. He proves that if each $r_i(t)$ converges to a limit, then $r(t)$ converges to the lowest of these limits. In other words, if

$$\lim_{t\to\infty} r_i(t) = r_i^*$$

for each i, then

$$\lim_{t\to\infty} r(t) = \min\{r_1^*, \ldots, r_n^*\}. \tag{3}$$

Moreover, if each r_i is constant (i.e., each D_i is exponential), then $r(t)$ declines *monotonically* to this limit (Weitzman, 1998).[1]

Example 1. Suppose each D_i is exponential, so that $r_i(t) = r_i$ is constant. Then the results in Weitzman (1998) imply that $r(t)$

[1] In fact, this is true more generally – see Weitzman (1998, footnote 6).

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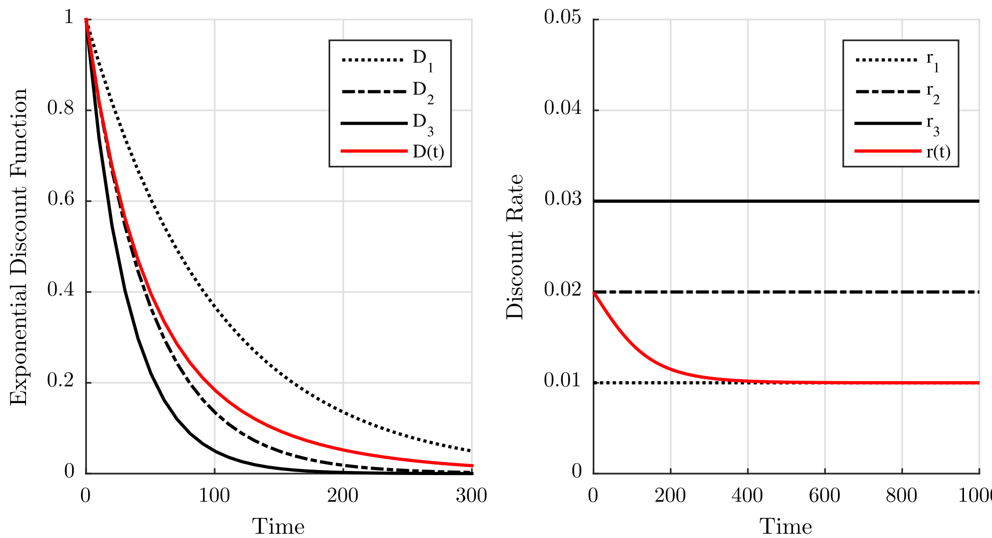

Fig. 1. Exponential Discount Functions.

declines monotonically with $\lim_{t\to\infty} r(t) = \min_i r_i$. Fig. 1 illustrates for the case $n = 3$, $r_1 = 0.01$, $r_2 = 0.02$, $r_3 = 0.03$ and $p_1 = p_2 = p_3 = 1/3$.

Weitzman's result may be interpreted as saying that the certainty equivalent discount function (2) behaves locally as an exponential discount function with discount rate (3) when discounting outcomes in the far distant future. This result is most salient if the individual discount functions are themselves exponential, as in Example 1. However, many individuals do *not* discount exponentially (Frederick et al., 2002). If all the D_i functions are contained within some non-exponential class, it is natural to characterize the local asymptotic behaviour of (2) using a function from the same class. The next section considers the case of proportional hyperbolic functions.

2. Proportional hyperbolic discounting

It is well known (see, for example, Frederick et al. (2002) and Loewenstein and Prelec (1992)) that individuals typically have intertemporal preferences that are inconsistent with exponential discounting but compatible with time preference rates that *decrease* with t. Such preferences exhibit *decreasing impatience (DI)*.[2] The necessity of accommodating DI has made hyperbolic discounting a significant tool in behavioural economics. Several types of hyperbolic discount functions have been introduced, including quasi-hyperbolic (Laibson, 1997; Phelps and Pollak, 1968), proportional hyperbolic (Harvey, 1995; Mazur, 2001), and generalized hyperbolic (Al-Nowaihi and Dhami, 2006; Loewenstein and Prelec, 1992).

In this section we assume that each D_i has the proportional hyperbolic form

$$D_i(t) = \frac{1}{1 + h_i t}$$

where parameter $h_i > 0$ is the *hyperbolic discount rate*. Note that

$$r_i(t) = -\frac{D_i'(t)}{D_i(t)} = \frac{h_i}{1 + h_i t}$$

so $r_i(t)$ decreases over time.

Prelec (2004) defines a local index of DI, analogous to the Arrow–Pratt index of absolute risk aversion for preferences over lotteries. For the proportional hyperbolic discount function D_i this index is

$$I_i(t) = -\frac{r_i'(t)}{r_i(t)} = \frac{h_i}{1 + h_i t}.$$

Prelec (2004) also introduces a comparative notion of DI, analogous to the notion of "more risk averse than" for lottery preferences, and shows that D_i is "more decreasingly impatient than" D_j if $I_i(t) \geq I_j(t)$ for all t. For proportional hyperbolic discount functions, we observe that $I_i(t) \geq I_j(t)$ iff $h_i \geq h_j$. The hyperbolic discount rate therefore determines how rapidly impatience diminishes.

We henceforth assume that the discount functions have been indexed such that $h_1 > h_2 > \cdots > h_n$, so D_1 exhibits the most rapidly diminishing impatience and D_n the least. Nevertheless, the limiting behaviour of these discount functions is indistinguishable through Weitzman's lens, since

$$r_i^* = \lim_{t\to\infty} \frac{h_i}{1 + h_i t} = 0$$

for each i. In other words, the limit of the local *exponential* discount rate is the same for each discount function, reflecting the fact that hyperbolic functions decline more slowly than exponentials for large t. Weitzman's result is therefore not very informative for this scenario.

Instead, we should like to have a local *hyperbolic* approximation to the certainty equivalent discount function (2). We follow Weitzman's example and define the *local (or instantaneous) hyperbolic discount rate*, $h(t)$, as follows:

$$D(t) = \frac{1}{1 + h(t)t} \quad \Leftrightarrow \quad h(t) = \left(\frac{1}{D(t)} - 1\right)\frac{1}{t}. \tag{4}$$

Note that $h(t)$ is constant if (and only if) D has the proportional hyperbolic form.

The question we wish to address is the following: *How does $h(t)$ behave as $t \to \infty$?* Theorems 1 and 2, which are proved in the Appendix, provide the answer. In order to state the second of these results, we remind the reader that the *weighted harmonic mean* of non-negative values $x_1, x_2, \ldots, x_n$ with non-negative weights $a_1, a_2, \ldots, a_n$ satisfying $a_1 + \cdots + a_n = 1$ is

$$H(x_1, a_1; \ldots; x_n, a_n) = \left(\sum_{i=1}^{n} \frac{a_i}{x_i}\right)^{-1}.$$

It is well-known that the weighted harmonic mean is smaller than the corresponding weighted arithmetic mean (i.e., expected value).

[2] See Prelec (2004) for a formal definition.