Economics Letters 155 (2017) 149–153

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

Linear–quadratic term structure models for negative euro area yields

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HIGHLIGHTS

- Four factor linear–quadratic models (LQTSM) fit negative Euro yields well.
- As in Euroland, in LQTSM short yields can be negative, but not the longest yields.
- LQTSM outperform four factor quadratic models that permit negative yields.
- Quadratic models that permit negative yields outperform affine Gaussian models.
- Quadratic models that rule out negative yields seem no longer adequate.

ARTICLE INFO

Article history:
Received 6 February 2017
Received in revised form 20 March 2017
Accepted 23 March 2017
Available online 31 March 2017

JEL classification:
G12
G13

Keywords:
Linear–quadratic term structure models
Quadratic models
Discrete time
Negative yields
Extended Kalman Filter

ABSTRACT

Four factor linear–quadratic models (LQTSM) fit negative Euro yields well, as short yields can be negative, but not the longest yields. LQTSM outperform four factor quadratic models that permit negative yields, which in turn outperform affine Gaussian models.

1. Introduction

Until few years ago negative nominal Government bond yields were almost unknown and economic theory ruled them out, since "investors can always hold their cash". Affine Gaussian term structure models (AGTSM) were regularly criticised because they permitted negative bond yields. After the 2008 crisis, academic research even concentrated on term structure models with a zero lower bound, i.e. models that could rule out negative yields while at the same time matching very low observed yields. However as of early 2017 negative yields have been observed for extended periods in Japan, Euroland and elsewhere, which somewhat vindicates AGTSM. Now we need models that can match at least slightly negative yields. In this spirit this paper tests linear–quadratic term structure models (LQTSM) whereby the short rate may turn negative, but not the longest yields, since the central tendency of the factors driving the short rate is a quadratic non-negative function of Gaussian factors. This paper also considers quadratic term structure models (QTSM) that do permit negative yields. All tests use four factor models and AAA-rated euro area Government bond yields of maturities up to 30 years.

The evidence shows that LQTSM perform much better than AGTSM and much better than QTSM that rule out negative yields. LQTSM can match the moderately negative yields for maturities up to ten years and at the same time the higher positive yields observed for the longest maturities up to thirty years. Also specifications of QTSM that permit negative yields perform better than AGTSM and better than "classic" QTSM that rule out negative yields, but slightly worse than LQTSM.

This paper tests a LQTSM that builds on Realdon (2011, 2016), who proposed discrete time versions of the continuous time linear–quadratic pricing model of Cheng and Scaillet (2007). The next section presents the pricing models and another section illustrates their empirical performance.

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http://dx.doi.org/10.1016/j.econlet.2017.03.029

2. Discrete time linear–quadratic term structure models (LQTSM)

This section presents LQTSM in discrete time. Discrete time implies Gaussian conditional transition density for the factors, which helps model estimation through Kalman Filters. LQTSM encompass linear Gaussian models and quadratic models as special cases. Let $V_{n,t}$ be the time t value of a zero coupon bond with n trading days to maturity, thus the bond matures on trading day $t+n$. Each time step is equal to $\Delta = \frac{1}{261}$ as we observe about 261 trading days per year. r_t is the continuously compounded risk-free interest rate for the trading day $[t, t+1]$, therefore $V_{1,t} = e^{-\Delta \cdot r_t}$ and $r_t = \frac{-\ln V_{1,t}}{\Delta}$. Following Realdon (2011) we further assume

$$r_t = \beta'\mathbf{x}_t + \mathbf{x}_t'\Psi x_t + \delta'\mathbf{y}_t \tag{1}$$

$$\mathbf{x}_t = \left(x_{1,t}, \dots, x_{m,t}\right)', \mathbf{y}_t = \left(y_{1,t}, \dots, y_{p,t}\right)' \tag{2}$$

$$\mathbf{x}_{t+1} - \mathbf{x}_t = \phi\left(\mu - \mathbf{x}_t\right) + \Sigma\xi^{\mathbb{Q}}_{t+1} \tag{3}$$

$$\mathbf{x}_{t+1} - \mathbf{x}_t = \phi^*\left(\mu^* - \mathbf{x}_t\right) + \Sigma\xi_{t+1} \tag{4}$$

$$\mathbf{y}_{t+1} - \mathbf{y}_t = \phi_y\left(\mu_y + diag\left(\mathbf{x}_t'\mathbf{L}_j\mathbf{x}_t\right)\cdot \mathbf{1}_{p\times 1} - \mathbf{y}_t\right) + \left(\Sigma_{yx}, \Sigma_y\right)\begin{pmatrix}\xi^{\mathbb{Q}}_{t+1}\\ \xi^{\mathbb{Q}}_{y,t+1}\end{pmatrix} \tag{5}$$

$$\mathbf{y}_{t+1} - \mathbf{y}_t = \phi_y^*\left(\mu_y^* + diag\left(\mathbf{x}_t'\mathbf{L}_j\mathbf{x}_t\right)\cdot \mathbf{1}_{p\times 1} - \mathbf{y}_t\right) + \left(\Sigma_{yx}, \Sigma_y\right)\begin{pmatrix}\xi_{t+1}\\ \xi_{y,t+1}\end{pmatrix} \tag{6}$$

$$\left(\xi^{\mathbb{Q}\prime}_{t+1}, \xi^{\mathbb{Q}\prime}_{y,t+1}\right)' \backsim N\left(\mathbf{0}_{m+p}, \mathbf{I}_{m+p}\right), \left(\xi'_{t+1}, \xi'_{y,t+1}\right)' \backsim N\left(\mathbf{0}_{m+p}, \mathbf{I}_{m+p}\right) \tag{7}$$

$$\xi^{\mathbb{Q}}_{t+1} = \left(\varepsilon^{\mathbb{Q}}_{1,t+1}, \dots, \varepsilon^{\mathbb{Q}}_{m,t+1}\right)', \xi_{t+1} = \left(\varepsilon_{1,t+1}, \dots, \varepsilon_{m,t+1}\right)' \tag{8}$$

$$\xi^{\mathbb{Q}}_{y,t+1} = \left(\varepsilon^{\mathbb{Q}}_{y,1,t+1}, \dots, \varepsilon^{\mathbb{Q}}_{y,p,t+1}\right)', \xi_{y,t+1} = \left(\varepsilon_{y,1,t+1}, \dots, \varepsilon_{y,p,t+1}\right)' \tag{9}$$

$$\Sigma = \mathbf{S}\sqrt{\Delta}, \Sigma_{yx} = \mathbf{S}_{yx}\sqrt{\Delta}, \Sigma_y = \mathbf{S}_y\sqrt{\Delta} \tag{10}$$

$$\phi = \Delta\cdot\kappa, \phi^* = \Delta\cdot\kappa^*, \phi_y = \Delta\cdot\kappa_y, \phi_y^* = \Delta\cdot\kappa_y^*, \tag{11}$$

$$V_{n,t} = \exp\left(A_n + \mathbf{B}_n'\mathbf{x}_t + \mathbf{x}_t'\mathbf{C}_n\mathbf{x}_t + \mathbf{D}_n'\mathbf{y}_t\right) \tag{12}$$

where: $diag\left(\mathbf{x}_t'\mathbf{L}_j\mathbf{x}_t\right)$ is a diagonal matrix whose jth diagonal entry is $\mathbf{x}_t'\mathbf{L}_j\mathbf{x}_t$; $\mathbf{L}_j$ is a $m\times m$ matrix and $\mathbf{1}_{p\times 1}$ is a $p\times 1$ matrix whose elements are all equal to 1; $\mathbf{x}_t, \mu, \mu^*, \xi^{\mathbb{Q}}_{t+1}, \xi_{t+1}, \mathbf{B}_n$; are $m\times 1$ vectors; $\Psi, \phi, \phi^*, \kappa, \kappa^*, \mathbf{C}_n, \Sigma, \mathbf{S}, \mathbf{L}_j$ are $m\times m$ matrices; r_t, A_n, are scalars; $\Sigma_{yx}, \mathbf{S}_{yx}$ are $p\times m$ matrices; $\mathbf{y}_t, \mu_y, \varepsilon^{\mathbb{Q}}_{y,t+1}, \varepsilon_{y,t+1}, \mathbf{D}_n$ are $p\times 1$; vectors; $\phi_y, \phi_y^*, \kappa_y, \kappa_y^*, \Sigma_y, \mathbf{S}_y$ are $p\times p$ matrices; $N\left(\mathbf{0}_{(m+p)\times 1}, \mathbf{I}_{m+p}\right)$ denotes the multivariate normal density with mean $\mathbf{0}_{(m+p)\times 1}$ and covariance matrix $\mathbf{I}_{m+p}$; $\mathbf{0}_{(m+p)\times 1}$ is a $(m+p)\times 1$ matrix of zeros; $\mathbf{I}_{m+p}$ is the $(m+p)\times(m+p)$ identity matrix; $\varepsilon_{1,t+1}, \dots, \varepsilon_{m,t+1}, \varepsilon_{y,1,t+1}, \dots, \varepsilon_{y,p,t+1}$ and $\varepsilon^{\mathbb{Q}}_{1,t+1}, \dots, \varepsilon^{\mathbb{Q}}_{m,t+1}, \varepsilon^{\mathbb{Q}}_{y,1,t+1}, \dots, \varepsilon^{\mathbb{Q}}_{y,p,t+1}$ are scalar Gaussian random shocks respectively in the real and risk-neutral measures. The processes of the factors $\mathbf{x}$ and $\mathbf{y}$ are specified under both the real measure and the risk-neutral measure $\mathbb{Q}$. The discount bond value $V_{n,t}$ is exponential linear in $\mathbf{y}_t$ and exponential linear–quadratic in $\mathbf{x}_t$. This discrete time linear–quadratic model is a special case of Realdon (2011), whereby we can compute $A_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$ appearing in (12) as

$$A_n = A_{n-1} + \mathbf{B}_{n-1}'\phi\mu + \mathbf{D}_{n-1}'\phi_y\mu_y + (\phi\mu)'\mathbf{C}_{n-1}\phi\mu + \ln\frac{|\gamma|}{abs\,|\Sigma|} + \frac{1}{2}\left(\mathbf{G}_{n-1} + \mathbf{D}_{n-1}'\Sigma_{yx}'\Sigma^{-1}\right)\gamma\gamma'\left(\mathbf{G}_{n-1} + \mathbf{D}_{n-1}'\Sigma_{yx}'\Sigma^{-1}\right)' + \frac{1}{2}\mathbf{D}_{n-1}'\Sigma_y\Sigma_y'\mathbf{D}_{n-1} \tag{13}$$

$$\mathbf{B}_n' = -\Delta\beta' + \mathbf{B}_{n-1}'\left(\mathbf{I}_m - \phi\right) + 2(\phi\mu)'\mathbf{C}_{n-1}\left(\mathbf{I}_m - \phi\right) + \mathbf{D}_{n-1}'\phi_{yx} + 2\left(\mathbf{G}_{n-1} + \mathbf{D}_{n-1}'\Sigma_{yx}'\Sigma^{-1}\right)\gamma\gamma'\mathbf{C}_{n-1}'\left(\mathbf{I}_m - \phi\right) \tag{14}$$

$$\mathbf{C}_n = -\Delta\Psi + (\mathbf{I}_m - \phi)'\mathbf{C}_{n-1}\left(\mathbf{I}_m - \phi\right) + \sum_{j=1}^{p} D_{j,n-1}\cdot\mathbf{L}_j + 2(\mathbf{I}_m - \phi)'\mathbf{C}_{n-1}\gamma\gamma'\mathbf{C}_{n-1}'\left(\mathbf{I}_m - \phi\right) \tag{15}$$

$$\mathbf{D}_n = -\Delta\delta + \left(I - \phi_y\right)'\mathbf{D}_{n-1} \tag{16}$$

$$\mathbf{G}_{n-1} = \mathbf{B}_{n-1}' + 2(\phi\mu)'\mathbf{C}_{n-1}$$

$$\gamma = \left(\left(\Sigma\Sigma'\right)^{-1} - 2\mathbf{C}_{n-1}\right)^{-1/2}$$

$$A_0 = 0,\quad \mathbf{B}_0 = \mathbf{0}_{m\times 1},\quad \mathbf{C}_0 = \mathbf{0}_{m\times m},\ \mathbf{D}_0 = \mathbf{0}_{p\times 1}.$$

We compute $A_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$ with 261 steps per year and $\Delta = 1/261$. The stochastic factors are latent. In this paper we focus on the following special cases.

2.1. The linear–quadratic model LQ2.2

LQ2.2 is such that $\delta = \mathbf{1}_{2\times 1}$, $\Psi = \mathbf{0}_{2\times 2}$ and $\beta = \mathbf{0}_{2\times 1}$ so that $r_t = y_{1,t} + y_{2,t}$ and

$$\mathbf{S} = \begin{pmatrix}\sigma_1 & 0\\ \rho_{12}\cdot\sigma_2 & \sqrt{1-\rho_{12}^2\cdot\sigma_2}\end{pmatrix}, \mu = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \mu^* = \begin{pmatrix}\mu_1^*\\ \mu_2^*\end{pmatrix},$$

$$\kappa = \begin{pmatrix}\kappa_{1,1} & 0\\ \kappa_{2,1} & \kappa_{2,2}\end{pmatrix}, \kappa^* = \begin{pmatrix}\kappa_{1,1}^* & 0\\ \kappa_{2,1}^* & \kappa_{2,2}^*\end{pmatrix}$$

$$\kappa_y = \begin{pmatrix}\kappa_{y_1} & 0\\ 0 & \kappa_{y_2}\end{pmatrix}, \mu_y = \mu_y^* = \begin{pmatrix}0\\ 0\end{pmatrix},$$

$$diag\left(\mathbf{x}_t'\mathbf{L}_j\mathbf{x}_t\right)\cdot\mathbf{1}_{2\times 1} = \begin{pmatrix}x_{1,t}^2\\ x_{2,t}^2\end{pmatrix}$$

$$\left(\mathbf{S}_{yx}, \mathbf{S}_y\right) = \begin{pmatrix}\sigma_{y_1}\left(\rho_{1,y_1}, Q_{2,y_1}, \sqrt{1-\rho_{1,y_1}^2 - Q_{2,y_1}^2}, 0\right)\\ \sigma_{y_2}\left(\rho_{1,y_2}, Q_{2,y_2}, Q_{y_1,y_2}, \sqrt{1-\rho_{1,y_2}^2 - Q_{2,y_2}^2 - Q_{y_1,y_2}^2}\right)\end{pmatrix},$$

$$\begin{pmatrix}\xi^{\mathbb{Q}}_{t+1}\\ \xi^{\mathbb{Q}}_{y,t+1}\end{pmatrix} = \begin{pmatrix}\varepsilon^{\mathbb{Q}}_{1,t+1}\\ \varepsilon^{\mathbb{Q}}_{2,t+1}\\ \varepsilon^{\mathbb{Q}}_{y,1,t+1}\\ \varepsilon^{\mathbb{Q}}_{y,2,t+1}\end{pmatrix}$$

$$Q_{2,y_1} = \frac{\rho_{2,y_1} - \rho_{12}\cdot\rho_{1,y_1}}{\sqrt{1-\rho_{12}^2}}, Q_{2,y_2} = \frac{\rho_{2,y_2} - \rho_{12}\cdot\rho_{1,y_2}}{\sqrt{1-\rho_{12}^2}},$$

$$Q_{y_1,y_2} = \frac{\rho_{y_1,y_2} - \rho_{1,y_1}\rho_{1,y_2} - Q_{2,y_1}\cdot Q_{2,y_2}}{\sqrt{1-\rho_{1,y_1}^2 - Q_{2,y_1}^2}}.$$

ρ_{12} is the conditional correlation between $x_{1,t+1}$ and $x_{2,t+1}$, ρ_{1,y_1} between $x_{1,t+1}$ and $y_{1,t+1}$, ρ_{2,y_1} between $x_{2,t+1}$ and $y_{1,t+1}$ and ρ_{y_1,y_2} has similar meaning. LQ2.2 is of interest since the factors $y_{1,t}$ and $y_{2,t}$ drive the short interest rate r_t, while longer term yields are driven by $x_{1,t}$ and $x_{2,t}$. Therefore short term and long term yields can move quite independently. $y_{1,t}$ tends to revert toward the level $x_{1,t}^2$ and $y_{2,t}$ tends to revert toward the level $x_{1,t}^2$, therefore long term yields tend to be positive when $\kappa_{y_1}, \kappa_{y_2} > 0$ and $\mu_{y_1}, \mu_{y_2} \geq 0$. In LQ2.2 we impose $\mu_y = \mu_y^* = \mathbf{0}_{2\times 1}$, $\mu_1 = \mu_2$, $\mu_1^* = \mu_2^*$ for the sake of parsimony and without much loss. The model can match very low and even negative short term yields and at the same time also positive long term yields. The main contribution of this paper is to provide evidence that LQ2.2 outperforms the other affine or quadratic models.